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ALGEBRA

THEORETICAL AND PRACTICAL

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PREFACE.

The present Treatise contains all the subjects in theory and practice usually comprehended in an elementary work on Algebra. It has been a special object to explain, as clearly and concisely as possible, the principles of the science, to illustrate them fully by appropriate examples, and to prescribe a sufficient number of exercises for solution by the student, in order to impress the principles on his memory, and enable him to acquire sufficient skill in Algebraic Computation.

In preparing this new edition, the objects aimed at have been to secure perfect accuracy in all the answers, to supply defects, and to adapt the work to the present advanced state of the science. For this purpose, the questions have all been solved several times, and numerous Theorems have been given which were not contained in the former editions. Multiplication by Detached Coefficients, Synthetic Division, new applications of Undetermined Coefficients, Elimination by Cross Multiplication, and a new Treatise on the General Solution of the Higher Equations, have been introduced; together with a great number of additional exercises on the most useful kinds of Equations. In order that the work might be contained within due limits, the Solution of Indeterminate Equations of the second degree has been omitted, as being of little or no practical utility; and the Notes and Appendix have been incorporated in the body of the work.

The method adopted in this Treatise is, to state the rule concisely and clearly; to illustrate it by appropriate gradational examples; to demonstrate the rule; and to prescribe a series of exercises. By this means the transition is very gradual from the more simple to the more complex parts of the subject.

After the student has made himself familiar with the subjects contained in this Treatise, he will be well prepared for entering successfully on the study of the application of Algebra to Analytical Trigonometry, Analytical Geometry, and the Differential and Integral Calculus.

There are some articles near the beginning of the Treatise which should be omitted in the first reading:—The discussion of Insulated Negative Quantities in Addition and Subtraction; the examples with Literal Coefficients in Addition, Subtraction, and Multiplication, which ought to be postponed till the student has studied the third case of Division; and also those subjects marked with an asterisk in the Contents.

JAMES PRYDE.

EDINBURGH, *May* 1852.

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ALGEBRA.

PRELIMINARY DEFINITIONS AND PRINCIPLES.

1. ALGEBRA is that branch of Mathematics which teaches the method of performing calculations by means of letters, which represent numbers or quantities, and symbols which indicate the operations to be performed on them.

2. *Quantity* is anything that is capable of increase or diminution.

3. *Measurable* or *mathematical quantity* is that which is capable of being measured.

Thus, lines, surfaces, time, are measurable quantities, and therefore proper objects of mathematical calculation.

4. The *unit of measure* of any quantity is some portion of the same kind of quantity assumed as the standard of measurement.

Thus, to ascertain the length of a line, some unit of measure, as a foot, is assumed, and the number of feet in the line is called its length.

5. The *numerical value* of a quantity is expressed by the number of times that the assumed unit of measure is contained in it.

Thus, if a foot be assumed as the unit of measure of a line which contains 20 feet, its numerical value is 20. The same line may have different numerical values according to the assumed unit of measure.

6. When the kind of units of which a number is composed is not mentioned, the number is said to be *abstract*, but when the denomination is specified, it is said to be *concrete*.

Thus, 20 feet is a *concrete* number, the kind of units being stated, but 20 is an *abstract* number, as the kind is not mentioned.

7. Since quantities may be represented by numbers, and are always supposed to be so in algebra, the letters denote numbers; and in the theory of algebra, the letters have no particular

numerical value, so that they represent any number or any quantity—that is, general or *abstract* quantity.

8. *Theoretical* algebra investigates the properties of abstract quantity and the rules of algebraical computation; and *practical* algebra is concerned in the solution of questions, in which the given quantities either have particular values, or may have such values assigned.

9. In algebra, as in geometry, there are two kinds of propositions—namely, *theorems* and *problems.*

10. In a *theorem*, it is proposed to demonstrate some stated relation or property of numbers or abstract quantities.

11. In a *problem*, the object is to find the value of some unknown numbers or quantities by means of given relations existing between them and other known numbers or quantities.

DEFINITIONS OF QUANTITIES AND SIGNS OF OPERATION.

12. *Known quantities* are those whose numerical values are given or supposed to be known; and *unknown quantities* are those whose numerical values are not known.

Known quantities are commonly represented by the first letters of the alphabet, as a, b, c; and unknown quantities by the last letters, as x, y, z.

13. The sign of *equality*, $=$, read *equal to*, denotes that the quantities between which it is placed are equal, and the whole expression is called an *Equation.*

Thus, $a = b$ denotes that a is equal to b.

14. The sign of *addition*, $+$, read *plus*, denotes that the quantities between which it is placed are to be added together.

Thus, $a + b$ means that b is to be added to a. If $a = 4$, and $b = 3$, then $a + b = 4 + 3 = 7$.

15. The sign of *subtraction*, $-$, read *minus*, denotes that the quantity after it is to be subtracted from the quantity before it.

Thus, $a - b$ means that b is to be subtracted from a. If $a = 8$, and $b = 5$, then $a - b = 8 - 5 = 3$.

16. The signs $+$ and $-$ are called *the* signs, the former is called the *positive*, and the latter the *negative* sign; and they are said to be *contrary* or *opposite.*

17. Quantities having positive signs, are said to be *positive;* and those having negative signs, are called *negative.* When a quantity has *no sign* prefixed, the positive is always understood.

18. Quantities that have the same sign, are said to have *like* signs; and those that have opposite signs, are said to have *unlike* signs.

19. The sign of difference, $\sim$, read *difference between*, denotes the difference of any two quantities between which it is placed.

Thus, $a \sim b$ or $b \sim a$, denotes the difference between a and b. If $a = 8$, and $b = 6$, $a \sim b = 8 \sim 6 = 2$; or $b \sim a = 6 \sim 8 = 2$.

20. The *double* or *ambiguous* sign, $\pm$, read *plus* or *minus*, and

$\pm$, read *sum* or *difference*, are sometimes placed between two quantities: the former denotes the sum of the quantities, or the excess of the first above the second; and the latter denotes the sum or difference of the two quantities.

Thus, $a \pm b$ means the sum of a and b, or the excess of a above b. So $a \underset{\sim}{+} b$ denotes the sum of a and b, or their difference. If $a = 8$, and $b = 5$, then $a \pm b = 8 \pm 5 = 13$ or 3; and $a \underset{\sim}{+} b = 8 \underset{\sim}{+} 5 = 13$ or 3.

21. The sign of multiplication, $\times$, read *into* or *multiplied by*, means that the quantities between which it is placed are to be multiplied together.

Instead of the sign $\times$, a *dot* or point is often used.

The product of two quantities is also expressed by writing them in *close succession*, as the letters of a word.

The last is the most common notation. Thus, $a \times b$, $a \,.\, b$, or ab, all denote that a is to be multiplied by b. If $a = 5$, and $b = 4$, $a \times b$, $a \,.\, b$, or $ab = 5 \times 4 = 20$.

22. Quantities that are to be multiplied together, are called *factors*. The *continued product* of several factors means that the product of the first and second is to be multiplied by the third, this product by the fourth, and so on (146). And the *product* of more than two factors means their *continued* product; which is represented by placing the sign of multiplication or a dot between the succeeding factors, or more commonly by writing the letters which denote the factors as the letters of a word.

Thus, the continued product of a, b, and c is expressed by $a \times b \times c$, or $a \,.\, b \,.\, c$, or more commonly by abc. If $a = 3$, $b = 5$, $c = 4$, then $abc = 3 \times 5 \times 4 = 15 \times 4 = 60$.

23. The sign of *division*, $\div$, read *divided by*, denotes that the quantity preceding it is to be divided by the quantity following it. The division of two quantities is more commonly represented by placing the dividend as the numerator, and the divisor as the denominator of a fraction, which forms an *algebraic* fraction.

Thus, $a \div b$ or $\frac{a}{b}$, means that a is to be divided by b. If

$a = 20$, and $b = 5$, then $a \div b$ or $\frac{a}{b} = 20 \div 5$, or $\frac{20}{5} = 4$.

24. The following terms used in arithmetic — namely, sum, minuend, subtrahend, remainder; multiplicand, multiplier, product; and dividend, divisor, and quotient — have the same meaning in algebra.

25. The sign of *majority*, $>$, read *is greater than*, denotes that the quantity before it exceeds the quantity after it.

Thus, $a > b$ means that a exceeds b. If $a = 8$, and $b = 5$, then $8 > 5$.

26. The sign of *minority*, $<$, read *is less than*, denotes that the quantity preceding it is less than the quantity following it.

Thus $a \angle b$ means that a is less than b. If $a = 5$ and $b = 6$, then $5 \angle 6$.

27. The sign ∞ or $\frac{1}{0}$, denotes a quantity greater than any that can be assigned; that is, a quantity indefinitely great, or *infinity*.

28. The *numeral coefficient* of a quantity is a number prefixed to it, which shews how often the quantity is to be taken.

Thus, $4a$ means a taken 4 times; $\frac{2}{3}x$ means $\frac{2}{3}$ of x; so $5ax$ means that the product ax is to be taken 5 times. If $a = 2$, and $x = 3$, then $5ax = 5 \times 2 \times 3 = 5 \times 6 = 30$.

29. The *literal coefficient* of a quantity is a letter by which it is multiplied.

Thus, in the quantity az, a may be considered to be a coefficient of z, or z a coefficient of a. Quantities having literal coefficients are commonly considered to be unknown, and the coefficients to be known, quantities.

30. The *coefficient* of a quantity may consist of a number together with a literal part.

Thus, in $5abx$, the quantity $5ab$ may be considered to be the coefficient of x. If $a = 2$, and $b = 3$, then $5ab = 5 \times 2 \times 3 = 5 \times 6 = 30$, and $5abx = 30x$.

When no numeral coefficient is prefixed to a quantity, unity is understood to be its coefficient. Thus, the coefficient of x, or of ax, or of abx, is 1; that is, $a = 1a$, and $ax = 1ax$.

31. A *power* of a quantity is the result arising from the multiplication of the quantity into itself one or more times.

When the quantity is repeated twice as a factor, the product is called its *square*, or *second* power; when it is repeated three times, the *cube*, or *third* power; when four times, the *fourth* power; and so on.

Instead of repeating the same quantity as a factor, a small figure is placed over it, to point out the number of times that the quantity is repeated. This figure is called the *exponent*; and hence a small 2 is used to denote the square; 3, the cube; 4, the fourth power; 5, the fifth power; and so on.

Thus the square or second power of	x is $= xx$,	and is expressed by	x^2
cube or third ...	x is $= xxx$,	...	x^3
the fourth ...	x is $= xxxx$,	...	x^4
the fifth ...	x is $= xxxxx$,	...	x^5
.....	.....	.....	

and generally the nth power $= xxx \ldots$ where x is supposed to be repeated n times, and is expressed by x^n.

When a letter has no exponent, it is considered to be the *first* or *simple* power of the quantity, and *unity* is considered to be its exponent; thus, x or x^1 is the first power of x.

32. To *involve* a quantity to any power, is to find that power of the quantity.

33. The *square root* of any proposed quantity is that quantity whose square, or second power, gives the proposed quantity. The *cube root* is that quantity whose cube is the proposed quantity; and so on. The nth root is that quantity whose nth power is the proposed quantity.

34. To *extract* any root of a quantity, is to find that root.

35. The *radical sign*, $\sqrt{\ }$, placed before a quantity, indicates that some root of it is to be extracted; and a small figure placed over the sign, called the *exponent* of the root, shews what root is to be extracted. Roots are also indicated by fractional exponents. Thus, the square root of a is indicated either by $\sqrt{a}$ or $a^{\frac{1}{2}}$; the cube root, by $\sqrt[3]{a}$ or $a^{\frac{1}{3}}$; the fourth root, by $\sqrt[4]{a}$ or $a^{\frac{1}{4}}$; and the nth root, by $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$.

36. A *simple* quantity consists of a single letter, or the product of two or more letters, or powers of letters, with or without a coefficient.

Thus, a, $5a$, $-a^2x$, $3ax^2y$, are simple quantities.

37. A *compound* quantity consists of two or more simple quantities called its *terms*, which are connected by the signs plus and minus.

Thus, $a+b$, $3a-2x$, $2a-3x^2y+cz$, are compound quantities.

38. A simple quantity is also called a *monomial;* a compound quantity, consisting of two terms, is called a *binomial;* one of three terms, a *trinomial;* of four terms, a *quadrinomial;* and of more than four terms, a *multinomial* or *polynomial.* A binomial, whose second term is negative, is called a *residual quantity.*

39. When any operation is to be performed on a compound quantity, it is expressed by enclosing the quantity in parentheses ( ), and inserting the sign of the operation.

Thus, $-(a-2ax)$ means that $a-2ax$ is to be subtracted; $5(3a-2x)$ means that $3a-2x$ is to be multiplied by 5; $(3a-2x)(a-b)$ means that $3a-2x$ is to be multiplied by $a-b$; $(a+x)\div(a-x)$ means that $a+x$ is to be divided by $a-x$; $(a-x)^2$ means that $a-x$ is to be squared; and $\sqrt[3]{(a^2-x^2)}$ means that the cube root of a^2-x^2 is to be extracted.

A *vinculum* ——— is sometimes used instead of parentheses.

Thus, $\overline{a-x}\,.\,y^2$ means that $a-x$ is to be multiplied by y^2.

A perpendicular *bar* | is sometimes used to denote that a compound quantity is to be multiplied by another quantity.

Thus, $\left.\begin{array}{r} a \\ -2c \\ +d \end{array}\right| x^3$ has the same meaning as $(a-2c+d)x^3$.

40. The *dimension* or *degree* of any simple quantity or term is

the number of first or simple powers of factors of which it is composed, and is indicated by the sum of its exponents.

Thus, ax is of the second dimension, for the sum of its exponents is two; $3a^2x^3$ is of the fifth dimension, for the sum of its exponents is five; and so on.

41. Quantities that are exactly the same, except in their signs and coefficients, are called *like* quantities; other quantities are said to be *unlike*.

Thus, $2a^2x^3$, $-5a^2x^3$, are like quantities; and $3x^2y$, $4xy^2z$, are unlike.

42. An *insulated* negative quantity is one with a negative sign, which is considered to exist unconnected with a positive quantity.

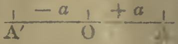
Thus, $-a$, when it is not supposed to be subtracted from any positive quantity, is called an insulated negative quantity, and, for the sake of distinction, it may be enclosed in brackets, thus $[-a]$.

43. The sign of an insulated negative quantity is called a sign of *relation* or *affection;* whereas in the case of an ordinary negative or subtractive quantity, it is the sign of the *operation* of subtraction.

44. When a negative quantity is considered as *insulated*, it is taken in a *sense directly the opposite* of that in which it would be understood if it were a positive quantity.

Thus, if $+a$ represent a number of miles in one direction, then $[-a]$ indicates an equal number in an opposite direction. So if $+a$ mean a number of pounds of property, $[-a]$ means an equal amount of debt. If $+10$ means 10 miles west or of *westing*, $[-10]$ means 10 miles east or of *easting*, or of *negative westing*. So if $+a$ means a number of men who *pay* away money, $[-a]$ means the same number who *receive* money. The expression that $-a$ is a less than nothing merely means that it is an insulated negative quantity.

Since all algebraic quantities represent numbers (7), and numbers may be represented by lines, $+a$ may be represented by the line OA, and $[-a]$ by the equal line OA′ in the opposite direction; and thus may all insulated negative quantities be represented, even when they denote abstract numbers.



NUMERICAL EVALUATION OF ALGEBRAIC EXPRESSIONS.

45. When numerical values are assigned to algebraic quantities, the numerical value of any algebraic expression containing them may be found by substituting for the letters their equivalent numerals, and then performing on these numbers the operations indicated by the algebraic signs contained in the expression.

EXAMPLES.

1. Find the numerical value of the expression $a - 2b + 3c^2$, when $a = 3$, $b = 4$, and $c = 6$.

$a - 2b + 3c^2 = 3 - 2 \times 4 + 3 \times 6^2 = 3 - 8 + 108 = 111 - 8 = 103$.

46. Any other numerical values may be assigned to the letters a, b, and c, and the expression converted in like manner into its numerical value, which will generally be different for different values of the letters.

2. Find the value of $a^4 + 2(a - x)(a + x) - 4xy^3$, when $a = 3$, $x = 2$, and $y = 5$.

This quantity $= 3^4 + 2(3 - 2)(3 + 2) - 4 \times 2 \times 5^3 = 81 + 2 \times 1 \times 5 - 4 \times 2 \times 125 = 81 + 10 - 1000 = 91 - 1000 = -909$.

3. Find the value of $(a^2 - x^2)(a^2 + x^2) - \dfrac{3(a - x)^2}{a + x} + 3\sqrt{ax}$, when $a = 9$ and $x = 4$.

It becomes $= (81 - 16)(81 + 16) - \dfrac{3 \times 5^2}{9 + 4} + 3\sqrt{9 \times 4} = 65 \times 97 - \dfrac{3 \times 25}{13} + 3\sqrt{36} = 6305 - \dfrac{75}{13} + 3 \times 6 = 6305 + 18 - 5\dfrac{10}{13} = 6323 - 5\dfrac{10}{13} = 6317\dfrac{3}{13}$.

4. For any value whatever of x and y, the expression $(x + y) \times (x - y)$ is equal to $x^2 - y^2$ or $(x - y)(x + y) = x^2 - y^2$.

Let $x = 5$ and $y = 3$,
then $(x + y)(x - y) = (5 + 3)(5 - 3) = 8 \times 2 = 16$,
and $x^2 - y^2 = 5^2 - 3^2 = 25 - 9 = 16$, the same result as before.

Let $x = 6$ and $y = 4$,
then $(x + y)(x - y) = (6 + 4)(6 - 4) = 10 \times 2 = 20$,
and $x^2 - y^2 = 6^2 - 4^2 = 36 - 16 = 20$.

EXERCISES.

In the following exercises, the values given to the different letters are, $a = 4$, $b = 3$, $c = 5$, $d = 10$, $x = 2$, and $y = 6$. Any other values, however, may be assigned, but then the numerical values of the expressions will generally be different.

1. Find the numerical value of the expression $4a$, $= 16$.
2. Find the value of $-3ax$, ... $= -24$.
3. Find the value of $6a^2x$, ... $= 192$.
4. Find the value of $3a^2 - 2x$, ... $= 44$.

5. Find the value of $3(a + x)$, . . . $=$ 18.
6. Find the value of $3a^2 - 4x^3$, . . . $=$ 16.
7. Find the value of $a^2 + 3ax - x^2$, . . $=$ 36.
8. Find the value of $x^2 - 3(a + x)(a - x) + 2by$, $=$ 4.
9. Find the value of $3ax - \dfrac{2(a - x)}{3(a + x)} - 4\sqrt{2ax}$, $=$ $7\frac{7}{9}$.
10. Find the value of $\dfrac{2ax^2}{(a - x)^2} - 6\sqrt{ax^2}$, . . $= -16$.

In the following exercises the first and second sides will always give the same numerical value, if the same value be given to the letters on each side: verify this.

11. $2(a - x)^3 = 2a^2 - 4ax + 2x^2$.
12. $3(a + x)(a - x) = 3a^2 - 3x^2$.
13. $\dfrac{4(a^2 - x^2)}{a + x} = 4(a - x)$.
14. $\dfrac{a^4 - x^4}{a - x} = a^3 + a^2x + ax^2 + x^3$.

AXIOMS.

The following axioms are employed in algebraical as well as in geometrical reasoning:—

47. If equals be added to equals, the sums are equal.
48. If equals be taken from equals, the remainders are equal.
49. If equals be multiplied by equals, the products are equal.
50. If equals be divided by equals, the quotients are equal.
51. If equal quantities be involved to the same powers, these powers are equal.
52. If the same roots of equal quantities be extracted, these roots are equal.

ADDITION.

CASE I. TO ADD LIKE QUANTITIES WITH LIKE SIGNS.

53. RULE. To the sum of the coefficients prefix the common sign, and annex the common literal part.

EXAMPLES.

1.	2.	3.
$3a$	$-6xy$	$2ax^2 - 3by^3 + 4$
$2a$	$-xy$	$ax^2 - 2by^3 + 2$
a	$-4xy$	$5ax^2 - by^3 + 8$
$5a$	$-3xy$	$8ax^2 - 7by^3 + 5$
Sum $= 11a$	$-14xy$	$16ax^2 - 13by^3 + 19$

In the first example, suppose $a = 2$, then $3a = 3 \times 2 = 6$, $2a = 2 \times 2 = 4$, $a = 2$, $5a = 5 \times 2 = 10$, and the sum of these values is $6 + 4 + 2 + 10 = 22$.

But the sum 22 is much more easily found from the algebraic sum $11a$, for $11a = 11 \times 2 = 22$.

In the second example, let $x = 3$ and $y = 2$, and the values of its terms will be

$$\begin{aligned} 6xy &= 6 \times 3 \times 2 = 36 \\ xy &= \quad\;\; 3 \times 2 = \;\;6 \\ 4xy &= 4 \times 3 \times 2 = 24 \\ 3xy &= 3 \times 3 \times 2 = 18 \end{aligned}$$

and the sum of these values is $= 84$

But this sum is very easily found from the algebraic expression $14xy$, for $14xy = 14 \times 3 \times 2 = 84$. And as these terms are subtractive, their sum is -84.

In the first example, let a mean 10 miles, then

$$\begin{aligned} 3a &= 3 \times 10 = 30 \text{ miles} \\ 2a &= 2 \times 10 = 20 \;\ldots \\ a &= \quad\;\; 10 = 10 \;\ldots \\ 5a &= 5 \times 10 = 50 \;\ldots \end{aligned}$$

and the sum is . . $= 110$ miles,
or the sum $= 11a = 11 \times 10 = 110$ ...

The exercises may all be thus numerically exemplified, by

assigning any values to the letters; but in each illustration of this kind the same numerical values of the letters must be retained throughout.

54. The principle of the rule may be thus explained:—

If $3a$ and $4a$ are to be added, the sum is evidently $7a$.

If $-5a$ and $-3a$ are to be added, or, in other words, if $5a$ and $3a$ are both to be taken away from some other quantity, there is to be taken away altogether just their sum, or $8a$; but $8a$ to be taken away is represented by $-8a$; that is, the sum of $-5a$ and $-3a$ is $=-8a$.

Let $5a$ and $3a$ be *insulated* negative quantities, or $[-5a]$ and $[-3a]$. Then, since these two quantities may represent lines measured towards the left from a given point (44), and these two lines together would be exactly equal to a single line $8a$ measured towards the left or $[-8a]$; therefore $[-5a]$ and $[-3a]$ taken together are equal to $[-8a]$.

If a represent 1 mile, and the left direction denote east, then $[-5a]=5$ miles east, and $[-3a]=3$ miles east; and, therefore, together, they are $=8$ miles east.

Or if a be £100, then $[-5a]$ is $=$ £500 of debt, $[-3a]$ is $=$ £300 of debt, and $[-8a]$ is $=$ £800 of debt; but 800 is the sum of 500 and 300; hence $[-8a]$ is the sum of $[-5a]$ and $[-3a]$.

EXERCISES.

1.
$$\begin{array}{r} 3axy \\ 2axy \\ axy \\ 5axy \\ \hline 11axy \end{array}$$

2.
$$\begin{array}{r} -\ 6bx^2 \\ -\ bx^2 \\ -\ 5bx^2 \\ -\ 12bx^2 \\ \hline -\ 24bx^2 \end{array}$$

3.
$$\begin{array}{rrr} 3ay & -\ 4bx^3 & +\ 7 \\ ay & -\ 5bx^3 & +\ 9 \\ 15ay & -\ 14bx^3 & +\ 15 \\ 3ay & -\ bx^3 & +\ 10 \\ \hline 22ay & -\ 24bx^3 & +\ 41 \end{array}$$

4.
$$\begin{array}{rrr} 4ax & -\ 3cz & +\ 5 \\ 8ax & -\ 5cz & +\ 3 \\ 10ax & -\ cz & +\ 9 \\ ax & -\ 15cz & +\ 1 \\ 2ax & -\ 4cz & +\ 4 \\ \hline 25ax & -\ 28cz & +\ 22 \end{array}$$

5.
$$\begin{array}{rrr} 5az & -\ 3c^2 & +\ 4by \\ 18az & -\ c^2 & +\ 5by \\ 2az & -\ 2c^2 & +\ by \\ 25az & -\ 4c^2 & +\ 8by \\ az & -\ c^2 & +\ 3by \\ \hline 51az & -\ 11c^2 & +\ 21by \end{array}$$

CASE II. TO ADD LIKE QUANTITIES WITH UNLIKE SIGNS.

55. RULE. Find the sum of the coefficients of the like positive quantities, and next the sum of the coefficients of the like negative quantities; subtract the less sum from the greater, and to the

difference annex the common literal part, prefixing the sign of the greater.

EXAMPLES.

1.	2.	3.
$2x$	$5xy^2$	$3\sqrt{ax^3} - 10cz^3 + 12$
$-\ 7x$	$-\ 2xy^2$	$-\ 2\sqrt{ax^3} + 5cz^3 - 3$
$12x$	$8xy^2$	$6\sqrt{ax^3} + 8cz^3 + 16$
x	$-\ 9xy^2$	$4\sqrt{ax^3} - 6cz^3 - 8$
$-\ 6x$	$-\ xy^2$	$-\ 14\sqrt{ax^3} + 3cz^3 - 24$
$2x$	$-\ 4xy^2$	$-\ 3\sqrt{ax^3} - 7$

4.

$$\begin{array}{r} -\ 2(x+y) - 3(a+c) + 5 \\ 5(x+y) + 4(a+c) - 8 \\ -\ 7(x+y) + 8(a+c) - 7 \\ 3(x+y) - 5(a+c) + 4 \\ \hline \text{Sum} = -\ (x+y) + 4(a+c) - 6 \end{array}$$

In the first example, the sum of the positive quantities is $15x$, and that of the negative is $-13x$, and the difference between 15 and 13 or $15 - 13 = 2$; hence the sum is $+2x$. In the second example, the sum of the positive quantities is $8xy^2$, and of the negative $-12xy^2$, and the difference between 12 and 8 is 4, the sign of the greater being $-$; hence the sum by the rule is $-4xy^2$. In the third example, the sum of the positive terms containing cz^3 is $16cz^3$, and the sum of the negative terms is $-16cz^3$; the difference between 16 and 16 is 0; hence the sum is 0. In this case the quantities are said to *destroy* each other.

The principle of the rule may be thus explained:—

1. When the quantities to be added are a positive and subtractive quantity of the same kind. Thus, the sum of $3a$, $-5a$, $8a$, and $-2a$, is evidently the same as $3a + 8a$, and $-5a$, and $-2a$, or $11a$ and $-7a$; but if $11a$ is to be added to some other quantity, and $7a$ to be subtracted, the result is evidently the same as if their difference or $4a$ be added; that is, the sum of $+11a$ and $-7a$ is $4a$.

But when the negative quantity exceeds the positive, it is to be considered as an insulated negative quantity, and its meaning being the opposite of that of a positive quantity, their sum is $-$ their difference, with the sign of the greater prefixed. Thus, if $8a$ mean property, and $[-10a]$ debt, their sum is $[-2a]$

or $2a$ of debt. Also, since $8a$ and $[-2a]$ give $6a$ and $8a$ and $[-3a]$ give only $5a$, the quantity $[-2a]$ is to be considered *greater* than $[-3a]$, just as 0 is considered greater than $-a$ (44.)

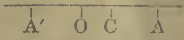
But to consider this case of addition under a more general point of view, let a be the positive and $[-b]$ the insulated negative quantity; then whatever quantity the former represents, the latter denotes the same kind of quantity in an opposite direction; that is, if a be a number of miles travelled west, $[-b]$ is a number travelled east, and the sum or $a + [-b]$ is $= a - b$ miles west, when $a > b$, or $a - b$ miles east, when $b > a$. The sum then is found in the same manner as that of a and $-b$, when b is a subtractive quantity.

56. Hence it appears that, to add a negative quantity, whether it be a common or an insulated one, is the same as to subtract an equal positive one.

57. The process of addition in this case is called *algebraic* addition, and the sum is called the *algebraic sum*, in order to distinguish them from arithmetical addition and arithmetical sum.

The object of the convention in the 44th definition is to identify the rules for the calculation of insulated negative quantities with those for common negative quantities; and, so far as addition is concerned, it appears that this object is accomplished. The above rule may be proved, by supposing a and b to represent each a number of miles, the one towards the west and the other towards the east; then the sum, that is, the whole number of miles from the original point of departure, is = the difference between a and b. But it is evident, from this peculiar relation, that if a and b represent any other like quantities taken in directly opposite senses or directions, their sum is = their difference taken in the sense to which the sign of the greater refers. Thus, if a is property and $[-b]$ debt or negative property, the net property is just $a - b$, that is, a and $[-b]$ taken together, or $a + [-b] = a - b$. If the difference between a and b be d, then when $a > b$, d is property, and when $a \angle b$, d is debt or negative property $= [-d]$.

All the possible cases that can occur may be represented by straight lines, as OA, OA′, reckoned in opposite directions from a point O. If



$OA = a$ miles travelled west, and $OA' = b$ miles travelled east, then the final distance from O is OC, if $AC = OA'$. For a point moving from O to A, or west, and then from A to C, or east, will finally be in the position C, but $OC = a - b$; that is, $a + [-b] = a - b$. If OA′ exceeded OA, the point C would lie towards the left of O, or $a - b$ would be so many miles east. Or OA may denote property and OA′ debt, then $OC =$ the net property; or, conversely, OA may denote debt, then OA′ denotes property, and thus $OC =$ the net debt.

EXERCISES.

1.

$$\begin{array}{r} 4a \\ -\ 3a \\ 8a \\ -\ 2a \\ \hline 7a \end{array}$$

2.

$$\begin{array}{r} 6axy \\ -\ 4axy \\ -\ 5axy \\ axy \\ \hline -\ 2axy \end{array}$$

3.

$$\begin{array}{r} 3az - 4by - 8 \\ -\ 2az + 5by + 6 \\ 5az + 6by - 7 \\ -\ 8az - 7by + 5 \\ \hline -\ 2az + 0 \quad - 4 \end{array}$$

4.

$$\begin{array}{r} 8ax - 3cz^2 \\ -\ 5ax - 5cz^2 \\ ax + 8cz^2 \\ -\ 3ax - 4cz^2 \\ 6ax + 7cz^2 \\ -\ 7ax - 3cz^2 \\ \hline 0 + 0 \end{array}$$

5.

$$\begin{array}{r} 12cy^5 - \ \ 4axz + \ \ 5mxy \\ 3cy^5 + \ \ axz - \ \ 3mxy \\ -\ 14cy^5 - 24axz - 15mxy \\ 16cy^5 + 36axz + 17mxy \\ 5cy^5 + \ \ 4axz - 13mxy \\ -\ 12cy^5 - 13axz - \ \ 8mxy \\ \hline 10cy^5 + \ \ 0 \quad - 17mxy \end{array}$$

6.

$$\begin{array}{r} 2ax^4 - 4xyz + \ \ 6 \\ -\ \ 3ax^4 + 4xyz - 12 \\ 5ax^4 - 8xyz + \ \ 8 \\ 25ax^4 - \ \ xyz + \ \ 9 \\ -\ \ ax^4 + 5xyz - 11 \\ \hline 28ax^4 - 4xyz + \ \ 0 \end{array}$$

7.

$$\begin{array}{r} 3(a + x) - 4(z + 2) \\ -\ \ 2(a + x) + 5(z + 2) \\ -\ \ 8(a + x) - 3(z + 2) \\ 12(a + x) - 2(z + 2) \\ -\ \ (a + x) + 5(z + 2) \\ \hline 4(a + x) + \ (z + 2) \end{array}$$

CASE III.—TO ADD QUANTITIES THAT ARE UNLIKE, OR PARTLY LIKE AND PARTLY UNLIKE.

58. RULE. Add the like quantities, by the preceding rules, and after them write the unlike quantities with their proper signs and coefficients.

EXAMPLES.

1.

$$\begin{array}{r} 3ax + \ \ yz \\ -\ \ 4yz + 5ax \\ -\ \ 3d \ + 2yz \\ 14ax + 5ax \\ -\ \ 7 \ \ - 3yz \\ \hline 27ax - 4yz - 3d - 7 \end{array}$$

2.

$$\begin{array}{r} 3ax^3 - 4cz^2 + \ \ 8 \\ 5m \ + 2ax^3 - 15 \\ 7 \ \ - 5ax^3 + \ \ 2cz^2 \\ -\ 2n \ + 2cz^2 + \ \ 6ax^3 \\ 3ax^3 - 2m \ + \ \ 3n \\ \hline 9ax^3 + 3m \ + \ \ \ n \end{array}$$

EXERCISES.

1.

$$\begin{array}{l} 5cy - 4az \\ -12 \quad + 8c \\ -\ 3cy + 5az \\ -\ 6 \quad - 2cy \\ \hline az - 18 + 8c \end{array}$$

2.

$$\begin{array}{l} 5(a + x) - 3cy \\ -4d \qquad + 24 \\ -8cy \qquad - 2(a + x) \\ -3(a + x) + 15cy \\ \hline 4cy - 4d + 24 \end{array}$$

3.

$$\begin{array}{l} 3xy - 5a + 6c \\ -\ 3m + 5 + xy \\ 12 \quad - 2m + 7 \\ 5m + 2xy - 24 \\ \hline 6xy - 5a + 6c \end{array}$$

4.

$$\begin{array}{l} 3(x + y) - 4c + 6 \\ -14 \quad + 5z - 3ax \\ -\ ax - 5(x + y) + 8 \\ 8(x + y) + 5d \\ \hline 6(x + y) - 4c + 5z - 4ax + 5d \end{array}$$

SUBTRACTION.

59. Rule. Change the signs of the terms of the subtrahend, or conceive them to be changed, and then proceed as in addition.

In the following examples and exercises, the quantity in the upper line is the minuend, and that in the lower line is the subtrahend:—

EXAMPLES.

1.

$$\begin{array}{l} 3ax - 2y^2 \\ -5ax - 8y^2 \\ \hline 8ax + 6y^2 \end{array}$$

2.

$$\begin{array}{l} 4cz^2 - 8x^2y^3 + 18 - 5(x + y) \\ 8cz^2 + 5x^2y^3 + 12 + 8(x + y) \\ \hline -4cz^2 - 13x^2y^3 + 6 - 13(x + y) \end{array}$$

In the first example, by the rule, $-5ax$ becomes $+5ax$, then by adding $3ax$ and $5ax$, the sum is $= 8ax$. Also $-8y^2$, by the rule, becomes $+8y^2$, and consequently $8y^2$ and $-2y^2 = +6y^2$.

3.

$$\begin{array}{l} 3ax - 5y^2 + 10 \\ 8y^2 + 3ax - c \\ \hline -13y^2 + 10 \quad + c \end{array}$$

4.

$$\begin{array}{l} 15cx^2y^4 - 3z^2 + 8d - 12 \\ -8 + 5z^2 - 4 - 8z^2 + 3z \\ \hline 15cx^2y^4 + 8d - 3z \end{array}$$

In the fourth example, $+5z^2$ and $-8z^2$ become $-5z^2$ and

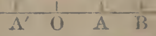
$+ 8z^2$; and these being added to $- 3z^2$, the sum is 0, for they destroy each other.

The proof of the rule of subtraction depends on the principle, that *the difference of two quantities is not altered by adding to, or subtracting from them both, the same quantity;* therefore write all the terms of the subtrahend after both minuend and subtrahend *with their signs changed* (which will be subtracting the positive from, and adding the negative terms of the subtrahend to, both), and their difference will not be changed; collect now the terms of the subtrahend, and their sum will be $= 0$; therefore the sum of the terms in the minuend will be the answer; but the sum is evidently the subtrahend with all its signs changed added to the minuend, which is identical with the rule.

60. When an insulated negative quantity as $[- c]$ is to be subtracted from any quantity as b, apply the principle of the proof given above, and thus we have, by adding c to both minuend and subtrahend $(b + c) - (c - c)$; but $c - c = 0$, therefore $b - [- c] = b + c$; hence, *to subtract a negative quantity, whether insulated or not, is the same as to add an equal positive quantity.*

61. The process of subtraction in this case gives the difference $(b + c)$, exceeding either of the given quantities; the difference is therefore called the *algebraic difference*, in order to distinguish it from arithmetical difference.

The distance between two points, A and B, on a line, having given their distances from a fixed point O in it, namely, OA and OB is $= (OB - OA) = AB$. Or if OA $= b$ and OB $= a$, then AB $= a - b$. When A is in the position A′ towards the left of O, the distance between A′ and B is $= (OB + OA')$, or the sum instead of the difference of the distances of A′ and B from O. But the rule established above makes A′B also in this case the difference between OB and OA′, for OA′ $= [- b]$, since it is measured in an opposite direction from OA, consequently, $a - [- b] = a + b$; that is, A′B is the *algebraic* difference between OB and OA′.



EXERCISES.

1.

$$\begin{array}{r} 3a - 2b \\ 5a - 3b \\ \hline -2a + b \end{array}$$

2.

$$\begin{array}{r} 6ax - 4y^2 + 3 - 5(c + z) \\ 4ax - 6y^2 + 2 - 7(c + z) \\ \hline 2ax + 2y^2 + 1 + 2(c + z) \end{array}$$

3.

$$\begin{array}{r} 2x^2 - 3a^2x^2 + 9 \\ x^2 + 5a^2x^2 - 3 \\ \hline x^2 - 8a^2x^2 + 12 \end{array}$$

4.

$$\begin{array}{r} 2(a + x) - 18 + 3(x + y) \\ -(a + x) + 12 + 15(x + y) \\ \hline 3(a + x) - 30 - 12(x + y) \end{array}$$

5.

$$\begin{array}{l} 4x^2y^3 - 5cz + 8m \\ -cz + 2x^2y^3 - 4cz \\ \hline 2x^2y^3 + 8m \end{array}$$

6.

$$\begin{array}{l} 6x^2 - 3cz - 18 + 6m \\ -12 + 7x^2 - 6 + 5 \\ \hline - x^2 - 3cz - 5 + 6m \end{array}$$

7.

$$\begin{array}{l} 3z^2 - 5y + 3axyz \\ - axyz + 18 - 3z^2 + 5n \\ \hline 6z^2 - 5y + 4axyz - 18 - 5n \end{array}$$

8.

$$\begin{array}{l} 15am^2n^3 - 11xyz + 3a \\ - 6xyz + 7 - 2a - 5xyz \\ \hline 15am^2n^3 + 5a - 7 \end{array}$$

ADDITION OF QUANTITIES WITH LITERAL COEFFICIENTS.

62. Rule. Collect the coefficients of the same quantity (58), enclosing them in parentheses, and after the sum write the common quantity.

In the following examples, x, y, z, are considered the quantities, and the other letters their coefficients :—

EXAMPLES.

1.

$$\begin{array}{l} ax + by^2z^2 \\ - bx + cy^2z^2 \\ dx - cy^2z^2 \\ \hline (a - b + d)x + (b + c - e)y^2z^2 \end{array}$$

2.

$$\begin{array}{r} - az - 4by - 6 \\ - 2cy - bz + 3 \\ bz + dy + 3az \\ \hline 2az - (4b + 2c - d)y - 3 \end{array}$$

In the second example, the coefficient of y is $-(4b + 2c - d)$ or $+(-4b - 2c + d)$, which are both identical, or $-4b - 2c + d$ (59).

EXERCISES.

1.

$$\begin{array}{l} 2ax^2 - 3bz^2 \\ - bx^2 + cz^2 \\ 3cx^2 - 2dz^2 \\ \hline \left\{\begin{array}{l}(2a - b + 3c)x^2 - \\ (3b - c + 2d)z^2\end{array}\right\} \end{array}$$

2.

$$\begin{array}{l} 2axy - 4cz^2 + 8 \\ - 5az^2 - 3bxy - 12 \\ 15 - 5cxy - 2cz^2 \\ \hline \left\{\begin{array}{l}(2a - 3b - 5c)xy \\ - (6c + 5a)z^2 + 11\end{array}\right\} \end{array}$$

3.

$$\begin{array}{l} 4cz^3 - 5y + 4z^2 \\ 3cy - 2cz^3 + dz^2 \\ - 5ay + 2ey - ez^2 \\ \hline \left\{\begin{array}{l}2cz^3 - (5 - 3c + 5a - 2e)y \\ + (4 + d - e)z^2\end{array}\right\} \end{array}$$

4.

$$\begin{array}{l} axyz + 3my^2 + 17 \\ - 5ny^2 + 2bxyz - 11 \\ 6m - 3cxyz - 6 \\ \hline \left\{\begin{array}{l}(a + 2b - 3c)xyz \\ + (3m - 5n)y^2 + 6m\end{array}\right\} \end{array}$$

SUBTRACTION OF QUANTITIES WITH LITERAL COEFFICIENTS.

63. RULE. Observe the former rule for subtraction (59), and the preceding one for the addition of quantities with literal coefficients (62).

EXAMPLES.

1.

$$\begin{array}{l} 3ax - 4by \\ 2cx + 3cy \\ \hline (3a - 2c)x - (4b + 3c)y \end{array}$$

2.

$$\begin{array}{l} 6ayz - 3bxy + 10 \\ -6cxy + 11 - 4cyz \\ \hline (6a + 4c)yz - (3b - 6c)xy - 1 \end{array}$$

In the first example, the term $2cx$ becomes $-2cx$, and then the coefficient of x is $3a - 2c$; and that of y likewise becomes $-4b - 3c$, or $-(4b + 3c)$.

EXERCISES.

1.

$$\begin{array}{l} ax^2 - by^2 \\ cx^2 - dy^2 \\ \hline (a - c)x^2 - (b - d)y^2 \end{array}$$

2.

$$\begin{array}{l} 2axyz - 3by^2 - 5ac \\ -2ac + 5axyz + 4cy^2 \\ \hline -3axyz - (3b + 4c)y^2 - 3ac \end{array}$$

3.

$$\begin{array}{l} 6bxy - 5cz^2 + 7 \\ -8 + 3cxy - 8dz^2 \\ \hline (6b - 3c)xy - (5c - 8d)z^2 + 15 \end{array}$$

4.

$$\begin{array}{l} 5ax - 4by + 3cz \\ -2dy + 3ex + 5by \\ \hline (5a - 3e)x - (9b - 2d)y + 3cz \end{array}$$

MULTIPLICATION.

CASE I.—TO MULTIPLY SIMPLE QUANTITIES WHEN THE MULTIPLICAND AND MULTIPLIER ARE DIFFERENT LETTERS.

64. RULE. Write the quantities in close succession, as in a word, and in alphabetic order, prefixing the product of the numerical coefficients to that of the quantities.

When the signs of the two quantities are like, the sign of the product is plus; and when the signs of the quantities are unlike, that of the product is minus. Hence *like signs give plus, and unlike signs give minus.*

EXAMPLES.

1. Multiply $4axy$ by $3bz$

$$4axy \times 3bz = 12abxyz$$

2. Multiply $5bx^2$ by $-3ayz^3$

$$5bx^2 \times -3ayz^3 = -15abx^2yz^3$$

3. Multiply $-\frac{3}{4}ax^4$ by $-\frac{4}{5}cy^5$

$$-\frac{3}{4}ax^4 \times -\frac{4}{5}cy^5 = \frac{3}{5}acx^4y^5$$

The reason of the rule for multiplying literal quantities is evident from (21); and that for the coefficients is also obvious from (28).

In regard to the sign of the product, it is evident that, when the signs of the two factors are plus, the product must be positive. For $a \times b$ means that a is to be taken b times; that is, a is to be added b times; and, therefore, the sum is positive.

Again, if $-a$ is to be multiplied by b, this merely means that $-a$ is to be repeated b times, or that a is to be subtracted b times; that is, a repeated b times is to be subtracted, or ab is to be subtracted; hence $-a \times b = -ab$.

If a is to be multiplied by $-b$, this merely means that a is to be subtracted b times; and hence the result must be the same as in the preceding case; that is, $-ab$.

If $-a$ is to be multiplied by $-b$, and $-a$ be considered as a common subtractive quantity, then $-a \times -b$ means that a taken subtractively is to be subtracted b times, which conveys no intelligible idea. But if $-a$ be an insulated negative quantity, then $[-a]$ subtracted once gives $+a$ (60); and therefore $[-a]$ subtracted b times gives $+ab$; that is, $[-a] \times -b = +ab$.

EXERCISES.

1. Multiply $3ab$ by $4cd$, ... $=$ $12abcd$.
2. ... $-8axz$ by $5by$, ... $=$ $-24abxyz$.
3. ... $16az^2$ by $-5bcxy^3$, . . . $=$ $-80abcxy^3z^2$.
4. ... $-5ax^4$ by $-4bcz^2$, . . . $=$ $20abcx^4z^2$.
5. ... $12cy^2$ by $-5ab^3x^5z^4$, . . $=$ $-60ab^3cx^5y^2z^4$.
6. ... $-3bx^5$ by $-12ac^2y^3$, . . $=$ $36abc^2x^5y^3$.

CASE II.—TO MULTIPLY SIMPLE QUANTITIES WHEN THE MULTIPLIER AND MULTIPLICAND CONSIST OF POWERS OF THE SAME LETTERS.

65. RULE. Add together the exponents of each letter for its exponent in the product.

EXAMPLES.

1. Multiply $3a^2x^5y^4$ by $2ax^2y$
$$3a^2x^5y^4 \times 2ax^2y = 6a^3x^7y^5$$
2. Multiply $5c^2y^3z$ by $-3cy^4z^2$
$$5c^2y^3z \times (-3cy^4z^2) = -15c^3y^7z^3$$
3. Multiply $2a^nx^{3m+1}$ by $3a^{2n}x^{5m-2}$
$$2a^nx^{3m+1} \times 3a^{2n}x^{5m-2} = 6a^{3n}x^{8m-1}$$

The rule may be proved thus.—
$$a^2 \times a^3 = aa \times aaa \ (31) = aaaaa = a^5.$$

Or, generally, $a^m \times a^n = aaa \ldots\ldots$, a being repeated m times, into $aaa \ldots\ldots$, a being repeated n times; that is, $aaaa \ldots\ldots$ a being repeated $(m+n)$ times; but the product $aaaa \ldots\ldots$ in which a is repeated $(m+n)$ times, or as often as is denoted by the sum of m and n, is represented by a^{m+n} (31); therefore $a^m \times a^n = a^{m+n}$.

EXERCISES.

1. Multiply $5a^4xz^3$ by $4a^3x^6z^2$, . . . $=$ $20a^7x^7z^5$.
2. ... $-7a^3by^2z$ by $3ab^2yz^2$. . . $=$ $-21a^4b^3y^3z^3$.
3. ... $-16c^2x^5z^4$ by $-5c^3xz$, . . $=$ $80c^5x^6z^5$.
4. ... $8a^{2n}x^my^{3r-3}$ by $-5a^{3n}x^my^{r+2}$, $=$ $-40a^{5n}x^{2m}y^{4r-1}$.

CASE III.—TO MULTIPLY SIMPLE QUANTITIES WHEN THE MULTIPLICAND AND MULTIPLIER CONSIST PARTLY OF DIFFERENT LETTERS, AND PARTLY OF POWERS OF THE SAME LETTER.

66. RULE. Find the product of the quantities denoted by different letters, by the rule in CASE I. (64); and that of the powers of the same quantity, by the rule in CASE II. (65), writing the letters in alphabetic order.

EXAMPLES.

1. Multiply $3ax^2y^4z$ by $2cy^2$

$$3ax^2y^4z \times 2cy^2 = 6acx^2y^6z$$

2. Multiply $5cx^5y^2$ by $-2cy^3z$

$$5cx^5y^2 \times (-2cy^3z) = -10c^2x^5y^5z$$

EXERCISES.

1.	Multiply $12a^5y^3$ by $5a^2x^2y^4$,	=	$60a^7x^2y^7$.
2.	... $-6a^4c^2z^5$ by $3c^6x^2z^4$,	=	$-18a^4c^8x^2z^9$.
3.	... $14ax^5y^4$ by $-2c^2x^3z^2$,	=	$-28ac^2x^8y^4z^2$.
4.	... $-15c^4e^3v^2$ by $-2aex^4$,	=	$30ac^4e^4x^4v^2$.
5.	... $3a^{2n}x^4y^m$ by $2a^{3n}y$,	=	$6a^{5n}x^4y^{m+1}$.
6.	... $3a^{2m}$ by $-2a^mx^n$,	=	$-6a^{3m}x^n$.

CASE IV.—TO MULTIPLY A COMPOUND MULTIPLICAND BY A SIMPLE MULTIPLIER.

67. Rule. Find separately the product of the multiplier, and each term of the multiplicand, by the preceding rules; then connect these partial products by their proper signs.

For if a quantity $a + b$ is to be multiplied by c, this is the same as adding $a + b$ to itself as often as there are units in c, or c times; which is evidently equal to $ac + bc$. So if $a - b$ is to be multiplied by c, this is the same as adding a to itself c times, and $-b$ to itself also c times, or it is $= ac - bc$. So if $a + b$ is to be multiplied by $-c$, this is just subtracting $a + b$, c times; or, what is equivalent, adding $-a - b$ to itself c times, which gives $-ac - bc$; and similarly, it may be shewn that $(a - b)(-c) = -ac + bc$; for $-b$ subtracted once gives $+b$, and subtracted c times, gives $+bc$, by the first case of multiplication.

EXAMPLES.

1. Multiply $2a^2 - 3ax + 5x^2$ by $3a^4$

$$\begin{array}{l} 2a^2 - 3ax + 5x^2 \\ 3a^4 \\ \hline 6a^6 - 9a^5x + 15a^4x^2 \end{array}$$

Here $2a^2$ is multiplied by $3a^4$, by the former case, which gives $6a^6$; then $-3ax$ is multiplied by $3a^4$, and then $5x^2$. It is usual in algebra to begin multiplication at the left hand, as was done also in addition and subtraction. It is sometimes convenient to represent the product in this case thus,

$$(2a^2 - 3ax + 5x^2)3a^4 = 6a^6 - 9a^5x + 15a^4x^2$$

2. Multiply $2ax - 3by^2$ by $3ab$.

$$\begin{array}{l} 2ax \quad - 3by^2 \\ 3ab \\ \hline 6a^2bx - 9ab^2y^2 \end{array}$$

3. ... $5a^2b^3x - 2by^3 + 3az^3$ by $- 2ay^2z^3$.

$$\begin{array}{l} \quad 5a^2b^3x \quad - 2by^3 \quad + 3az^3 \\ - \ 2ay^2z^3 \\ \hline - 10a^3b^3xy^2z^3 + 4aby^5z^3 - 6a^2y^2z^6 \end{array}$$

EXERCISES.

1. Multiply $2y^2 - 3xy + x^2$ by $4xy$, $= 8xy^3 - 12x^2y^2 + 4x^3y$.
2. ... $6ax^3 - 4a^2x^2 + 6a^3x$ by $2a^2x^2$,
$= 12a^3x^5 - 8a^4x^4 + 12a^5x^3$.
3. ... $3ax^2y - 5a^2xy^2 + 3x^5$ by $- 6x^2y$,
$= - 18ax^4y^2 + 30a^2x^3y^3 - 18x^7y$.
4. ... $2ax^2 - 3by^4 - 8x^3$ by $- 5abx$,
$= - 10a^2bx^3 + 15ab^2xy^4 + 40abx^4$.

CASE V.—TO MULTIPLY QUANTITIES WHOSE MULTIPLICAND AND MULTIPLIER ARE BOTH COMPOUND.

68. RULE. Multiply the multiplicand by each term of the multiplier, as in CASE IV., and collect the partial products by the rules of addition.

The reason of the rule is evident, from the explanation given in the last case.

EXAMPLES.

1.

Multiply $2a^2-3x^2$ by $3a^2-3ax$.

$$\begin{array}{l} 2a^2 - 3x^2 \\ 3a^2 - 3ax \\ \hline 6a^4 - 9a^2x^2 \\ \qquad - 6a^3x + 9ax^3 \\ \hline 6a^4 - 6a^3x - 9a^2x^2 + 9ax^3 \end{array}$$

2.

Multiply $a + x$ by $a + x$.

$$\begin{array}{l} a + x \\ a + x \\ \hline a^2 + ax \\ \qquad ax + x^2 \\ \hline a^2 + 2ax + x^2 \end{array}$$

The first partial product (ex. 1), that is, the first line of the

product, is obtained by multiplying the multiplicand by $3a^2$, and the second line by multiplying by $-3ax$; and lastly, these two partial products are added. The first term of each partial product is commonly placed under the term used as multiplier.

3.

Multiply $a - x$ by $a - x$.

$$\begin{array}{l}
a - x \\
a - x \\
\hline
a^2 - ax \\
\quad - ax + x^2 \\
\hline
a^2 - 2ax + x^2
\end{array}$$

4.

Multiply $a + x$ by $a - x$.

$$\begin{array}{l}
a + x \\
a - x \\
\hline
a^2 + ax \\
\quad - ax - x^2 \\
\hline
a^2 - x^2
\end{array}$$

5.

Multiply $a^2 + ax + x^2$ by $a - x$.

$$\begin{array}{l}
a^2 + ax + x^2 \\
a - x \\
\hline
a^3 + a^2x + ax^2 \\
\quad - a^2x - ax^2 - x^3 \\
\hline
a^3 + 0 + 0 - x^3 \\
\text{or } a^3 - x^3
\end{array}$$

6.

Multiply $a^2 - ax + x^2$ by $a + x$.

$$\begin{array}{l}
a^2 - ax + x^2 \\
a + x \\
\hline
a^3 - a^2x + ax^2 \\
\quad + a^2x - ax^2 + x^3 \\
\hline
a^3 + 0 + 0 + x^3 \\
\text{or } a^3 + x^3
\end{array}$$

7.

Multiply $a^{2n} - 2a^nx^n + x^{2n}$ by $a^n + 2x^n$.

$$\begin{array}{l}
a^{2n} - 2a^nx^n + x^{2n} \\
a^n + 2x^n \\
\hline
a^{3n} - 2a^{2n}x^n + a^nx^{2n} \\
\quad + 2a^{2n}x^n - 4a^nx^{2n} + 2x^{3n} \\
\hline
a^{3n} - 3a^nx^{2n} + 2x^{3n}
\end{array}$$

8.

Multiply $a^n + a^{n-1}x + a^{n-2}x^2 + \ldots ax^{n-1} + x^n$ by $a - x$.

69. As n has no particular value, the intermediate terms of the multiplicand cannot be supplied till such a value be assigned to it; yet the manner in which these terms are formed is evident

the fourth term would be $a^{n-3}x^3$; the last but two would be a^2x^{n-2}, and so on for the other terms.

$$\begin{array}{l} a^n + a^{n-1}x + a^{n-2}x^2 + \dots ax^{n-1} + x^n \\ a - x \\ \hline a^{n+1} + a^nx + a^{n-1}x^2 + \dots \quad a^2x^{n-1} + ax^n \\ \qquad - a^nx - a^{n-1}x^2 - \dots - a^2x^{n-1} - ax^n - x^{n+1} \\ \hline a^{n+1} - x^{n+1}. \end{array}$$

In the two partial products, all the terms intervening between the first and last destroy each other, hence the product is $a^{n+1} - x^{n+1}$, whether n be an *odd* or an *even* number, and therefore whether $n + 1$ be *even* or *odd*. But the quantity $a^n - a^{n-1}x + a^{n-2}x^2 - \dots + x^n$ has its last term *positive* only when n is an *even* number, and being multiplied by $a + x$, it gives $a^{n+1} + x^{n+1}$ where $n + 1$ is of course an *odd* number.

When n is an *odd* number, the last term is *negative*; thus, $a^n - a^{n-1}x + a^{n-2}x^2 - \dots - x^n$; which being multiplied by $a + x$, it gives $a^{n+1} - x^{n+1}$, in which $n + 1$ is of course an *even* number.

By giving n particular numerical values, the corresponding products will be easily found by substituting these numbers for n in the above products.

Since a and x in these examples may stand for any quantities, they contain the proofs of the five following *theorems*, which it is important to remember, as a knowledge of them will very much facilitate the solution of many of the following exercises :—

THEOREM I. The square of the sum of two quantities is equal to the sum of their squares increased by twice their product Proved by example 2.

THEOREM II. The square of the difference of two quantities is equal to the sum of their squares diminished by twice their product. Proved by example 3.

THEOREM III. The product of the sum and difference of two quantities is equal to the difference of their squares. Proved by example 4.

THEOREM IV. The sum of the squares of two quantities increased by their product, and multiplied by their difference, is equal to the difference of their cubes. Proved by example 5.

THEOREM V. The sum of the squares of two quantities diminished by their product, and multiplied by their sum, is equal to the sum of their cubes. Proved by example 6.

EXERCISES.

1. Multiply $2a^2 - 4ax + 2x^2$ by $3a - 3x$,
$= 6a^3 - 18a^2x + 18ax^2 - 6x^3$.

2. ... $3a^4 + 3x^4$ by $2a^4 - 2x^4$, . . $= 6a^8 - 6x^8$.

3. ... $2a^3 + 2a^2x + 2ax^2 + 2x^3$ by $3a - 3x$, $= 6a^4 - 6x^4$

4. ... $5a^2 + 5ax + 5x^2$ by $2a^2 - 2ax$, $= 10a^4 - 10ax^3$.

5. ... $3x^2 + 3xy + 3y^2$ by $2x^2 - 2y^2$,
$= 6x^4 + 6x^3y - 6xy^3 - 6y^4$.

6. ... $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$ by $a^2 - 2ax + x^2$,
$= a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6$.

7. ... $5x + 4y$ by $3x - 2y$, . . $= 15x^2 + 2xy - 8y^2$.

8. ... $x^2 + xy - y^2$ by $x - y$, . $= x^3 - 2xy^2 + y^3$.

9. ... $2x + 4y$ by $2x - 4y$, . $= 4x^2 - 16y^2$.

10. ... $x^2 - xy + y^2$ by $x + y$, . $= x^3 + y^3$.

11. ... $a^2 - ax - 6x^2$ by $a + 4x$,
$= a^3 + 3a^2x - 10ax^2 - 24x^3$.

12. ... $6a^2 + 10ax - 4x^2$ by $4a - 2x$,
$= 24a^3 + 28a^2x - 36ax^2 + 8x^3$.

13. ... $a^{2n} + x^{2n}$ by $2a^{2n} - 2x^{2n}$, . . $= 2a^{4n} - 2x^{4n}$.

14. ... $a^{2n} + a^nx^n + x^{2n}$ by $2a^n - 2x^n$, . $= 2a^{3n} - 2x^{3n}$.

15. ... $a^{3n} - a^{2n}x^n + a^nx^{2n} - x^{3n}$ by $3a^n + 3x^n$,
$= 3a^{4n} - 3x^{4n}$.

16. ... $a^n - a^{n-1}x + a^{n-2}x^2 - \ldots + ax^{n-1} - x^n$ by $2a + 2x$,
$= 2a^{n+1} - 2x^{n+1}$.

If the numerical exponents in any of the preceding exercises be multiplied by n, examples with literal exponents will be formed, the products of which will be the same as those of the former with their exponents multiplied by n.

MULTIPLICATION BY DETACHED COEFFICIENTS.

70. When the powers of the letters both in the multiplicand and multiplier either increase or decrease uniformly in each term, the multiplication may be performed with the coefficients alone, and the letters afterwards supplied; for, the power of the letter in the first term of the product will be the sum of its powers in the first terms of the multiplicand and multiplier, and the powers in the

product will increase or decrease in the same manner as in each factor. If some of the powers of the letters be absent, their place must be supplied by writing ciphers for their coefficients; thus,

1. Multiply $x^4 + 2x^3 - 3x^2 + 4x + 2$ by $x^2 + 3x + 2$.

$$\begin{array}{l}
1 + 2 - 3 + 4 + 2 \text{ coefficients of the multiplicand.} \\
1 + 3 + 2 \quad \ldots \quad \ldots \text{ multiplier.} \\
\hline
1 + 2 - 3 + 4 + 2 \\
\quad\; 3 + 6 - 9 + 12 + 6 \\
\quad\quad\; 2 + 4 - 6 + 8 + 4 \\
\hline
1 + 5 + 5 - 1 + 8 + 14 + 4 \text{ coefficients of the product.}
\end{array}$$

And since x^4 (the first term of the multiplicand), multiplied by x^2 (the first term of the multiplier), gives x^6, the product is

$$x^6 + 5x^5 + 5x^4 - x^3 + 8x^2 + 14x + 4.$$

2. Multiply $a^3 + 2ab^2 + 3b^3$ by $a^2 - 3b^2$.

Here the coefficient of a^2 in the multiplicand and of a in the multiplier are both zero; hence, by the rule,

$$\begin{array}{l}
1 + 0 + 2 + 3 \\
1 + 0 - 3 \\
\hline
1 + 0 + 2 + 3 \\
\quad\quad - 3 - 0 - 6 - 9 \\
\hline
1 + 0 - 1 + 3 - 6 - 9
\end{array}$$

produces $\quad a^5 - a^3b^2 + 3a^2b^3 - 6ab^4 - 9b^5.$

The coefficient of a^4 in the product being zero causes that term to disappear.

The preceding exercises can all be done in this way, and should be wrought again as shewn above.

EXAMPLES WITH LITERAL COEFFICIENTS.

71. In these examples, x, y, z, are considered as the quantities, and the other letters as their coefficients.

1. Multiply $x^2 - ax + b$ by $x - c$.

$$\begin{array}{l}
x^2 - ax + b \\
x - c \\
\hline
x^3 - ax^2 + bx \\
\quad\; - cx^2 + acx - bc \\
\hline
x^3 - (a + c)x^2 + (b + ac)x - bc
\end{array}$$

2. Multiply $x^2 - axy + by^2$ by $x - cy$.

$$\begin{array}{l} x^2 - axy + by^2 \\ x - cy \\ \hline x^3 - ax^2y + bxy^2 \\ \quad - cx^2y + acxy^2 - bcy^3 \\ \hline x^3 - (a + c)x^2y + (b + ac)xy^2 - bcy^3 \end{array}$$

EXERCISES.

1. Multiply $x^3 - px^2 + qx$ by $x - r$,
$$= x^4 - (p + r)x^3 + (q + pr)x^2 - qrx.$$
2. ... $x^2 - ax + b$ by $x - 1$,
$$= x^3 - (a + 1)x^2 + (a + b)x - b.$$
3. ... $x^3 - ax^2 + bx - c$ by $x^2 - x + 1$,
$$= x^5 - (1 + a)x^4 + (1 + a + b)x^3 - (a + b + c)x^2 + (b + c)x - c.$$
4. Multiply $2ax - 3cy$ by $bx + dy$,
$$= 2abx^2 - (3bc - 2ad)xy - 3cdy^2.$$

72. If a compound quantity be arranged according to the descending powers of one of the letters; that is, with the highest power first, and the rest in order, this letter is called the *leading* quantity.

Thus, in $3x^4 - 2x^3 + x^2 - 4x + 5$, x is the leading quantity.

73. A compound quantity which contains the second dimension of the leading quantity, is a *quadratic*.

Thus, $x^2 - 4x + 3$ is a *quadratic* trinomial.

Since $(x + a)(x + b) = x^2 + (a + b)x + ab$, it follows that the product of two simple binomial factors is a quadratic trinomial, and the coefficient of the simple power of the leading quantity is the sum of the second terms of the binomial, and the third term is their product. Thus—

$$\begin{aligned} (x + a)(x - b) &= x^2 + (a - b)x - ab \\ (x - a)(x + b) &= x^2 + (b - a)x - ab \\ (x - a)(x - b) &= x^2 - (a + b)x + ab \end{aligned}$$

Hence, when any quadratic trinomial occurs, which is the product of two binomials, with integers for their second terms, they may generally be easily decomposed into their factors. This resolution is frequently of great advantage in reducing fractions to their simplest forms; and may also be employed in many cases for finding the values of the unknown quantity in Quadratic Equations.

EXAMPLES.

1. Decompose $x^2 + 7x + 10$.

In this example it is easily seen that $7 = 2 + 5$, and $10 = 2 \times 5$; therefore, .. $x^2 + 7x + 10 = (x + 2)(x + 5)$.

2. Decompose $x^2 - 3x - 40$.

Here $-3 = -8 + 5$, and $-40 = -8 \times 5$; therefore $x^2 - 3x - 40 = (x - 8)(x + 5)$.

EXERCISES.

1. Decompose $x^2 + 10x + 24$ into two simple binomial factors, $= (x + 6)(x + 4)$.
2. ... $x^2 + 9x + 20$ into two factors. $= (x + 5)(x + 4)$
3. ... $x^2 + x - 20$, . . $= (x + 5)(x - 4)$.
4. ... $x^2 - x - 20$, . $= (x - 5)(x + 4)$.
5. ... $x^2 + x - 2$, . . $= (x + 2)(x - 1)$.
6. ... $x^2 - 2x + 1$, $= (x - 1)(x - 1)$.
7. ... $x^2 - 13x + 40$, . . $= (x - 8)(x - 5)$.
8. ... $x^2 - x - 132$, . $= (x - 12)(x + 11)$.
9. ... $x^2 - 3x - 40$, . . $= (x - 8)(x + 5)$.
10. ... $x^2 - 5x - 104$, . $= (x - 13)(x + 8)$.
11. ... $x^2 + 24x + 135$, . . $= (x + 9)(x + 15)$.

DIVISION.

CASE I.—TO DIVIDE SIMPLE QUANTITIES WHEN THE DIVIDEND AND DIVISOR ARE DENOTED BY DIFFERENT LETTERS.

74. RULE. Write the letters of the dividend and of the divisor in the form of a fraction, making the divisor the denominator, and prefix the quotient of the coefficients with the proper sign.

When the divisor and dividend have *like* signs, the sign of the quotient is *plus;* and when the signs are *unlike*, that of the quotient is *minus*.

The quotient of the coefficients will be either a whole number or a vulgar fraction: for it is not usual in algebraic division to reduce this quotient to the form of a decimal fraction.

EXAMPLES.

1. Divide $12acy$ by $3bx$.

$$12acy \div 3bx = \frac{12acy}{3bx} = \frac{4acy}{bx} \text{ or } 4\frac{acy}{bx}.$$

When the coefficient of the dividend, divided by that of the divisor, does not give an integer for the quotient, the coefficients are treated as the terms of a vulgar fraction. Thus, if the coefficients of the divisor and dividend be 5 and 4, their quotient is expressed by $\frac{5}{4}$. If they are 12 and 8, it is expressed by $\frac{12}{8}$ or $\frac{3}{2}$, by dividing both by 4, as in reducing fractions to their simplest form.

2. Divide $4ax^3$ by $12by^2$.

$$4ax^3 \div 12by^2 = \frac{4ax^3}{12by^2} = \frac{1}{3} \times \frac{ax^3}{by^2} = \frac{ax^3}{3by^2}.$$

3. ... $-8cz^4$ by $12ax^2y^3$.

$$-8cz^4 \div 12ax^2y^3 = \frac{-8cz^4}{12ax^2y^3} = -\frac{2cz^4}{3ax^2y^3} \text{ or } -\frac{2}{3} \times \frac{cz^4}{ax^2y^3}.$$

In this example, the coefficient $\frac{8}{12}$ is reduced to $\frac{2}{3}$ by the rule for reducing numerical fractions to their simplest form.

4. Divide $-24ayz$ by $-16bx^2$.

$$-24ayz \div (-16bx^2) = \frac{-24ayz}{-16bx^2} = \frac{3ayz}{2bx^2}.$$

The rule for dividing the literal part is evident from (23). That the quotient of the coefficients is to be prefixed to that of the letters, is evident from the proof in (77). Since the dividend is = the product of the divisor by the quotient, it is evident, from the rule for the signs in multiplication, that when the divisor and dividend have like signs, that of the quotient will be *plus*; and when unlike, that of the quotient will be *minus*. Hence —Like signs give *plus*, and unlike give *minus*, as well in Division as in Multiplication.

EXERCISES.

1. Divide $8a^4x^2$ by $2by^2$, ... $= \frac{4a^4x^2}{by^2}$.
2. ... $-12by$ by $6axz$, ... $= -\frac{2by}{axz}$.
3. ... $8acx$ by $-6by$, ... $= -\frac{4acx}{3by}$.

4. Divide $-20cyz$ by $12ax$, . . . $= -\frac{5cyz}{3ax}$.

5. ... $84a^2x^3$ by $-96c^2z^2$, ... $= -\frac{7a^2x^3}{8c^2z^2}$.

6. ... $-124cz$ by $-31ay$, ... $= \frac{4cz}{ay}$.

CASE II.—TO DIVIDE SIMPLE QUANTITIES WHEN THE DIVIDEND AND DIVISOR CONSIST OF POWERS OF THE SAME LETTERS.

75. RULE. Write the divisor and dividend in the form of a fraction, as in the last case; then take the difference between the exponents of the powers of the same quantity; this will be the exponent of that quantity in the quotient, which must be placed above or below the line, according as the greater exponent belongs to the dividend or divisor.

If the same quantity occur in the divisor and dividend, it may be cancelled from both.

As in multiplying powers of the same quantity, the sum of their exponents is taken, so in dividing, their difference is taken.

EXAMPLES.

1. Divide a^4x^3 by a^2x^2.

$$a^4x^3 \div a^2x^2 = \frac{a^4x^3}{a^2x^2} = a^{4-2}x^{3-2} = a^2x.$$

2. ... $-4a^5x^3$ by $2a^3x$.

$$\frac{-4a^5x^3}{2a^3x} = -2a^2x^2.$$

3. ... $-8c^6x^5yz^3$ by $-12c^4x^3y^5z^4$.

$$\frac{-8c^6x^5yz^3}{-12c^4x^3y^5z^4} = \frac{2c^{6-4}x^{5-3}}{3y^{5-1}z^{4-3}} = \frac{2c^2x^2}{3y^4z}.$$

4. ... $-24a^4x^3y^2z$ by $20a^6xy^4z$.

$$\frac{-24a^4x^3y^2z}{20a^6xy^4z} = -\frac{6x^2}{5a^2y^2}.$$

Here z is cancelled.

5. Divide a^{m+1} by a^m.

$$\frac{a^{m+1}}{a^m} = a^{m+1-m} = a^1 = a.$$

6. ... $a^{m+2}x^{m+1}$ by $a^{m-1}x$.

$$\frac{a^{m+2}x^{m+1}}{a^{m-1}x} = a^{m+2-(m-1)}x^{m+1-1} = a^{m+2-m+1}x^m = a^3x^m.$$

7. Divide $4a^{m-1}$ by $6a^{m-3}$.

$$\frac{4a^{m-1}}{6a^{m-3}} = \frac{2a^{m-1-(m-3)}}{3} = \frac{2a^{m-1-m+3}}{3} = \frac{2a^2}{3}.$$

8. ... a^{5m-2} by a^{2m+1}.

$$\frac{a^{5m-2}}{a^{2m+1}} = a^{5m-2-(2m+1)} = a^{5m-2-2m-1} = a^{3m-3}, \text{ or } a^{3(m-1)}.$$

The rule may be proved thus:—

1. When the exponent of the quantity in the dividend exceeds that in the divisor.

76. The dividend is just the product of the divisor and quotient: and therefore if a^7 is to be divided by a^3, the quotient must evidently be a^4, or $\frac{a^7}{a^3} = a^4$; so $\frac{a^8}{a^5} = a^3$; and generally $\frac{a^{m+n}}{a^n} = a^m$, for $a^m \cdot a^n = a^{m+n}$. So $\frac{a^m}{a^n} = a^{m-n}$, for $a^{m-n} \cdot a^n = a^m$. In this last example, it is understood that $m > n$.

2. When the exponent of the quantity in the dividend is less than that in the divisor.

Before establishing the rule in this case, it will be necessary previously to prove this proposition:—

77. The divisor and dividend may be both multiplied or both divided by the same quantity without altering the quotient. For if $N =$ the dividend, $D =$ the divisor; and if m, n, and d, be such quantities that $N = mn$, and $D = md$, then $\frac{N}{D} = \frac{n}{d}$. For if $\frac{n}{d} = q$, then if n be multiplied by 2, the quotient will evidently be $2q$; but if when n is doubled, d be also doubled, then $2n \div 2d$ will give just the same quotient q; so $3n \div 3d$ will give the quotient q; and, generally, $mn \div md$ will give for the quotient the same value q; that is, $\frac{mn}{md} = q$, or $\frac{N}{D} = q$. Hence $\frac{N}{D} = \frac{n}{d}$. That is, if $\frac{n}{d} = q$, then $\frac{mn}{md}$ or $\frac{N}{D}$ is $= q$. And if $\frac{N}{D} = q$, then $\frac{n}{d}$ is also $= q$, where $n = \frac{N}{m}$ and $d = \frac{D}{m}$.

From this principle, it is evident that $\frac{a^4}{a^7} = \frac{1 \times a^4}{a^3 \times a^4} = \frac{1}{a^3}$; for the quotient of 1, divided by a^3, is the same as that of a^4 divided by a^7, since, if these two quantities be each divided by the same quantity a^4, the new dividend and divisor are 1 and a^3. So $\frac{a^5}{a^8} = \frac{1}{a^3}$; and, generally, $\frac{a^n}{a^{m+n}} = \frac{1 \times a^n}{a^m \times a^n} = \frac{1}{a^m}$.

78. The proof in the second case establishes the rule that—

If same quantity be contained in the divisor and dividend, it may be cancelled from both.

EXERCISES.

1. Divide x^8 by x^5, ... $= x^3$.
2. ... x^3 by x^{11}, ... $= \dfrac{1}{x^8}$.
3. ... $-12a^3z^2$ by $4a^3z$, ... $= -3z$.
4. ... $-16c^4y^4$ by $-8c^8y^2$, ... $= \dfrac{2y^2}{c^4}$.
5. ... $12a^2x^3z^5$ by $-16a^3x^2z^6$, . . . $= -\dfrac{3x}{4az}$.
6. ... $-24x^4y^3z^2$ by $32x^2y^4z$, ... $= -\dfrac{3x^2z}{4y}$.
7. ... $6a^{3m}$ by $3a^{2m}$, ... $= 2a^m$.
8. ... $8a^{m-1}$ by $4a^{m-3}$, ... $= 2a^2$.
9. ... $a^{2m}x^m$ by a^mx^{3m}, ... $= \dfrac{a^m}{x^{2m}}$.
10. ... $4a^{3m}x$ by $8a^{m+2}x^3$, ... $= \dfrac{a^{2m-2}}{2x^2}$, or $\dfrac{a^{2(m-1)}}{2x^2}$.

79. It is sometimes convenient, and it extends the meaning of exponent, to take another view of the division of powers of the same quantity. Instead of subtracting the less exponent from the greater, if the latter be taken from the former, the exponent of the quotient will be negative; thus,

$$a^4 \div a^6 = \frac{a^4}{a^6} = a^{4-6} = a^{-2}, \text{ but } \frac{a^4}{a^6} = \frac{1}{a^2};$$

and, similarly, it is evident that

$$a^6 \div a^{10} = \frac{a^6}{a^{10}} = a^{6-10} = a^{-4}, \text{ but } \frac{a^6}{a^{10}} = \frac{1}{a^4}:$$

in the former case $\dfrac{a^4}{a^6}$ is equal both to a^{-2} and to $\dfrac{1}{a^2}$; and in the latter case, $\dfrac{a^6}{a^{10}}$ is equal both to a^{-4} and to $\dfrac{1}{a^4}$; hence $a^{-2} = \dfrac{1}{a^2}$, and $a^{-4} = \dfrac{1}{a^4}$; and similarly, $\dfrac{a^{3m}}{a^{4m}} = a^{3m-4m} = a^{-m}$, or $\dfrac{a^{3m}}{a^{4m}} = \dfrac{1}{a^m}$: therefore, generally, $\dfrac{1}{a^m} = a^{-m}$.

And it might be similarly shewn that $a^m = \dfrac{1}{a^{-m}}$.

80. These negative exponents are obtained by a process different from that given in the rule—namely, by subtracting the greater exponent from the less; and as no definition was formerly given of quantities with negative exponents, they can have no meaning further than that they are merely another mode of expressing other quantities, the meaning of which is known. Thus, a^{-m} is merely another mode of expressing $\frac{1}{a^m}$; or a^{-2} is another expression for $\frac{1}{a^2}$ or $\frac{1}{aa}$; so $\frac{1}{a^{-m}}$ is another mode of expressing a^m; as $\frac{1}{a^{-2}}$ is another mode of expressing a^2 or aa. This mode of expressing quantities is sometimes found to be convenient. It also evidently proves that—*Any quantity may be taken from the divisor to the dividend, or conversely, by changing the sign of its exponent.*

If the rule be applied to the case of equal exponents, another remarkable mode of expressing unity is obtained. Thus,

$$\frac{a^3}{a^3} = a^{3-3} = a^0; \text{ but } \frac{a^3}{a^3} = 1. \quad \text{So } \frac{a^n}{a^n} = a^{n-n} = a^0; \text{ but } \frac{a^n}{a^n} = 1.$$

Therefore $a^0 = 1$; hence—*Any quantity with the exponent 0 is equal to unity.*

Thus,
$$\frac{a^6}{a^4} = \frac{a^{-4}}{a^{-6}}, \text{ or } = a^{6-4} = a^2,$$

or
$$\frac{a^6}{a^4} = a^6a^{-4} = a^{6-4} = a^2,$$

or
$$\frac{a^6}{a^4} = \frac{1}{a^4a^{-6}} = \frac{1}{a^{4-6}} = \frac{1}{a^{-2}} = a^2.$$

The preceding exercises may be solved by this method: thus, for the fourth,

$$\frac{-16c^6y^4}{-8c^8y^2} = 2c^{6-8}y^{4-2} = 2c^{-2}y^2 = \frac{2y^2}{c^2}.$$

The result here found, namely, $\frac{2y^2}{c^2}$, may be easily obtained, by dividing the divisor and dividend by $-8c^6y^2$, for they may be written thus,

$$\frac{-8c^6y^2 \times 2y^2}{-8c^6y^2 \times c^2},$$

and if the common factor $-8c^6y^2$ be cancelled, the quotient is $\frac{2y^2}{c^2}$, as above; hence—

81. The rule for this case of division may also be expressed in (78).

EXAMPLES.

1. Divide $12a^6x^4$ by $8a^4x^5$.

$$\frac{12a^6x^4}{8a^4x^5} = \frac{3a^2 \times 4a^4x^4}{2x \times 4a^4x^4} = \frac{3a^2}{2x}.$$

In this example, $4a^4x^4$ is a factor common to both divisor and dividend, which may be expressed thus,

$$12a^6x^4 = 4a^4x^4 \times 3a^2, \text{ and } 8a^4x^5 = 4a^4x^4 \times 2x;$$

and hence, expunging the common factor $4a^4x^4$, the quotient is found.

To illustrate this method, the former examples may all be solved by it.

CASE III.—TO DIVIDE SIMPLE QUANTITIES WHEN THE DIVIDEND AND DIVISOR CONSIST OF QUANTITIES DENOTED BY DIFFERENT LETTERS, AND OF POWERS OF THE SAME LETTER.

82. Rule. Divide the quantities denoted by the different letters by the rule for CASE I., and the powers of the same quantity by any of the rules in CASE II.

EXAMPLES.

1. Divide $12c^5dx^3z^4$ by $-8bx^2z^6$.

$$\frac{12c^5dx^3z^4}{-8bx^2z^6} = -\frac{3c^5dx}{2bz^2}, \text{ by cancelling } 4x^2z^4.$$

2. Divide $6a^3x^2$ by $5bxy^2$.

$$\frac{6a^3x^2}{5bxy^2} = \frac{6a^3x}{5by^2}, \text{ by dividing by the common factor } x.$$

3. Divide $-18x^4y^2z^3$ by $9a^3x^6yz^4$.

$$\frac{-18x^4y^2z^3}{9a^3x^6yz^4} = -\frac{2y}{a^3x^2z}, \text{ by dividing both terms by } 9x^4yz^3.$$

4. Divide $-40a^5x^3z^6$ by $-25a^2xy^5z^4$.

$$\frac{-40a^5x^3z^6}{-25a^2xy^5z^4} = \frac{8a^3x^2z^2}{5y^5}, \text{ by cancelling the common factor } -5a^2xz^4$$

from both terms, or by any of the other methods (74–80).

EXERCISES.

1. Divide $6a^4x^2$ by $-3a^2xy$, ... $= -\frac{2a^2x}{y}$.

2. ... $-12c^4z^6$ by $8a^2c^2z^2$, ... $= -\frac{3c^2z^4}{2a^2}$.

3. Divide $6a^4c^2x^5$ by $4c^3x^2$, ... $= \frac{3a^4x^3}{2c}$

4. ... $-9x^4y^3$ by $6x^5z$, ... $= -\frac{3y^3}{2xz}$

5. ... $-24c^6x^5y^4$ by $-18c^2y^4$, ... $= \frac{4c^4x^5}{3}$

6. ... $120a^4x^5y^6$ by $96a^2xy^2z$, ... $= \frac{5a^2x^4y^4}{4z}$

CASE IV.—TO DIVIDE A COMPOUND QUANTITY BY A SIMPLE ONE.

83. Rule. Divide each term of the dividend by the divisor as in the preceding rules; and then connect the quotients by their proper signs.

EXAMPLES.

1. Divide $12ax^2 - 8a^2xy$ by $4ax$.

$$\frac{12ax^2 - 8a^2xy}{4ax} = 3x - 2ay.$$

Here $12ax^2$, divided by $4ax$, gives $3x$, and $-8a^2xy$, divided by $4ax$, gives $-2ay$.

2. Divide $24x^2y^3 - 15a^4x^5$ by $18a^2x^2$.

$$\frac{24x^2y^3 - 15a^4x^5}{18a^2x^2} = \frac{4y^3}{3a^2} - \frac{5a^2x^3}{6}.$$

3. ... $6a^2x^2 - 8a^4x^4 + 10a^2x^4$ by $4a^2x^2$.

$$\frac{6a^2x^2 - 8a^4x^4 + 10a^2x^4}{4a^2x^2} = \frac{3}{2} - 2a^2x^2 + \frac{5x^2}{2}.$$

4. ... $-24x^3y^5 + 18a^2x^4 - 15a^5x^6$ by $-12a^2x^3$.

$$\frac{-24x^3y^5 + 18a^2x^4 - 15a^5x^6}{-12a^2x^3} = \frac{2y^5}{a^2} - \frac{3x}{2} + \frac{5a^3x^3}{4}.$$

5. ... $-12c^4x^2 - 8x^5y^3 + 7x^6z$ by $3ax$.

$$\frac{-12c^4x^2 - 8x^5y^3 + 7x^6z}{3ax} = -\frac{4c^4x}{a} - \frac{8x^4y^3}{3a} + \frac{7x^5z}{3a}.$$

6. ... $3a^{2m} - 5a^mx^m + 8x^{2m}$ by $2a^mx^m$.

$$\frac{3a^{2m} - 5a^mx^m + 8x^{2m}}{2a^mx^m} = \frac{3a^m}{2x^m} - \frac{5}{2} + \frac{4x^m}{a^m}.$$

7. ... $6a^{4m} - 8a^{2m}x^m + 3a^mx^{4m}$ by $2a^{2m}x^m$.

$$\frac{6a^{4m} - 8a^{2m}x^m + 3a^mx^{4m}}{2a^{2m}x^m} = \frac{3a^{2m}}{x^m} - 4 + \frac{3x^{3m}}{2a^m}.$$

8. Divide $2x^m - 3x^{m-1} + 5x^{m-2}$ by $3x^{m-2}$.

$$\frac{2x^m - 3x^{m-1} + 5x^{m-2}}{3x^{m-2}} = \frac{2x^2}{3} - x + \frac{5}{3}.$$

Here x^m, divided by x^{m-2}, gives $x^{m-(m-2)} = x^{m-m+2} = x^2$.

EXERCISES.

1. Divide $12a^2x^2 - 3ax^3 + 5a^2$ by $3a^2$, . $= 4x^2 - \frac{x^3}{a} + \frac{5}{3}$.
2. ... $-8a^6y^4 + 12a^4y^6 - 9a^2y^2$ by $5by^2$,
$= -\frac{8a^6y^2}{5b} + \frac{12a^4y^4}{5b} - \frac{9a^2}{5b}$.
3. ... $16a^5x^7 - 3a^2y^5$ by $4a^2x^4$, . $= 4a^3x^3 - \frac{3y^5}{4x^4}$.
4. ... $8a^2 - 24c^3y^5z^4$ by $-8a^2y^5$, . $= -\frac{1}{y^5} + \frac{3c^3z^4}{a^2}$.
5. ... $12a^3x^4 - 5a^6x^7y^5 + 3ay^4z^2$ by $-4a^4y^2$,
$= -\frac{3x^4}{ay^2} + \frac{5a^2x^7y^3}{4} - \frac{3y^2z^2}{4a^3}$.
6. ... $11a^4 - 12x^6y^2 + 13z^6$ by $3a^2z^2$,
$= \frac{11a^2}{3z^2} - \frac{4x^6y^2}{a^2z^2} + \frac{13z^4}{3a^2}$.
7. ... $3a^2x^4y^5z^2 - 6a^4y^3z + 3ax^4z^2$ by $9a^2x^5z^2$,
$= \frac{y^5}{3x} - \frac{2a^2y^3}{3x^5z} + \frac{1}{3ax}$.
8. ... $2x^2y^2 - 3x^2z^2 + 4y^2z^2$ by $5x^2y^2z^2$,
$= \frac{2}{5z^2} - \frac{3}{5y^2} + \frac{4}{5x^2}$.
9. ... $6a^{3m}x^{2m} - 4a^{2m}x^{3m}$ by $3a^mx^m$, $= 2a^{2m}x^m - \frac{4a^mx^{2m}}{3}$.
10. ... $3ax^m - 4bx^{m-1} + 3cx^{m-2}$ by $2x^{m-1}$,
$= \frac{3ax}{2} - 2b + \frac{3c}{2x}$.
11. ... $16x^m - 4x^{m-4} + 6x^{m-8}$ by $4x^{m-6}$,
$= 4x^6 - x^2 + \frac{3}{2x^2}$.
12. ... $4a^m - 3x^m$ by $2a^mx^m$, . . $= \frac{2}{x^m} - \frac{3}{2a^m}$.
13. ... $6a^{4m} - 6a^{2m}x^{2m}$ by $3a^{3m}x^m$, . $= \frac{2a^m}{x^m} - \frac{2x^m}{a^m}$.
14. ... $ax^{4m} - bx^{6m} + cx^{8m}$ by dx^{6m}, $= \frac{a}{dx^{2m}} - \frac{b}{d} + \frac{cx^{2m}}{d}$.

By applying this case, compound quantities may often be reduced to more convenient forms; thus,

$$ax^2 - bx^2 + cx^2$$

being first divided by x^2, which is evidently a common factor of all the terms, gives the quotient

$$a - b + c;$$

and this being again multiplied by x^2, the product must be the original quantity, which is therefore

$$= (a - b + c)x^2,$$

in which the multiplication is merely represented; thus giving a form of expression much more concise than the given one.

So $2ax^2y - 3bx^2y + 5cx^2y = (2a - 3b + 5c)x^2y$; x^2y being a common factor of all the terms.

Also, $3ax^2y^3 - 5mxy^2 + 6nx^3y = (3axy^2 - 5my + 6nx^2)xy$; xy being a common factor.

84. Hence, to make such transformations, the terms of the given quantity must be divided by any factor that is common to them all; and the quotient then represented as multiplied by this factor. When there is no common factor, the form of the given quantity cannot be thus altered.

EXERCISES.

1. Transform $2ax - 3cx$, ... $= (2a - 3c)x$.
2. ... $4mxy - 3nxy + pxy$, . $= (4m - 3n + p)xy$.
3. ... $2acx^2z - 5abxz$, . . $= (2cx - 5b)axz$.
4. ... $3ax^2yz^2 - 5bx^2y^3z^2 + 2cx^3yz^2$,
$= (3a - 5by^2 + 2cx)x^2yz^2$.

CASE V.—TO DIVIDE QUANTITIES WHOSE DIVISOR AND DIVIDEND ARE BOTH COMPOUND.

85. Rule. Arrange the divisor and dividend according to the descending powers of some quantity common to them both; that is, so that the term containing the highest power of this quantity may stand first, and the rest in numerical order. Divide the first term of the dividend by that of the divisor, the result is the first term of the quotient. Multiply the divisor by this term, and subtract the product from the dividend. Considering the remainder as a new dividend, arrange its terms, and proceed as before; and thus continue the process till there be no remainder, or till the exponent of the leading quantity in the remainder be less than its exponent in the first term of the divisor.

EXAMPLES.

1. Divide $4a^2 - 8ax + 4x^2$ by $2a - 2x$.

$$\begin{array}{l} 2a - 2x)4a^2 - 8ax + 4x^2(2a - 2x \\ \qquad\quad\;\; 4a^2 - 4ax \\ \hline \qquad\qquad\quad - 4ax + 4x^2 \\ \qquad\qquad\quad - 4ax + 4x^2 \\ \hline \end{array}$$

Here the dividend as well as the divisor is arranged according to the powers of a; then $4a^2$, divided by $2a$, gives $2a$ for the first term of the quotient; and after multiplying the divisor by $2a$, and subtracting the product, the first term of the remainder is $-4ax$, which being divided by $2a$, gives $-2x$, the second and last term of the quotient; for there is no remainder after multiplying the divisor by $-2x$.

This process of division proceeds on the obvious principle that the dividend contains the divisor as often as its parts contain the divisor. Now, the dividend contains the divisor $2a$ times, with a remainder $-4ax + 4x^2$, and this remainder contains the divisor $-2x$ times, with no remainder; therefore the dividend contains the divisor exactly $(2a - 2x)$ times.

2. Divide $4a^4 - 4x^4$ by $2a^2 - 2x^2$.

$$\begin{array}{l} 2a^2 - 2x^2)4a^4 - 4x^4(2a^2 + 2x^2 \\ \qquad\qquad 4a^4 - 4a^2x^2 \\ \hline \qquad\qquad\qquad 4a^2x^2 - 4x^4 \\ \qquad\qquad\qquad 4a^2x^2 - 4x^4 \\ \hline \end{array}$$

3. ... $9a^6 - 9x^6$ by $3a^3 + 3x^3$.

$$\begin{array}{l} 3a^3 + 3x^3)9a^6 - 9x^6(3a^3 - 3x^3 \\ \qquad\qquad 9a^6 + 9a^3x^3 \\ \hline \qquad\qquad\quad - 9a^3x^3 - 9x^6 \\ \qquad\qquad\quad - 9a^3x^3 - 9x^6 \\ \hline \end{array}$$

4. ... $6a^6 - 6x^6$ by $2a^2 - 2x^2$.

$$\begin{array}{l} 2a^2 - 2x^2)6a^6 - 6x^6(3a^4 + 3a^2x^2 + 3x^4 \\ \qquad\qquad 6a^6 - 6a^4x^2 \\ \hline \qquad\qquad\qquad 6a^4x^2 - 6x^6 \\ \qquad\qquad\qquad 6a^4x^2 - 6a^2x^4 \\ \hline \qquad\qquad\qquad\qquad\quad 6a^2x^4 - 6x^6 \\ \qquad\qquad\qquad\qquad\quad 6a^2x^4 - 6x^6 \\ \hline \end{array}$$

When the divisor and dividend consist of a considerable number

of terms, it is convenient to write the divisor over the quotient, as in the following example:—

5. Divide $6a^4 - 15a^3x - 8ax^3 + 6x^4$ by $2a^2 - 3ax + x^2$.

$$\begin{array}{rrrrr|l}
6a^4 - 15a^3x - 8ax^3 + 6x^4 & \left\{\begin{array}{l}2a^2 - 3ax + x^2 \\ 3a^2 - 3ax - 6x^2\end{array}\right. \\
6a^4 - 9a^3x + 3a^2x^2 & \\
\hline
-6a^3x - 3a^2x^2 - 8ax^3 & \\
-6a^3x + 9a^2x^2 - 3ax^3 & \\
\hline
-12a^2x^2 - 5ax^3 + 6x^4 & \\
-12a^2x^2 + 18ax^3 - 6x^4 & \\
\hline
-23ax^3 + 12x^4 &
\end{array}$$

86. As the remainder contains only the first power of a, while the first term of the divisor contains its second power, the division cannot be carried further; therefore generally, when the exponent of the leading quantity in the first term of the remainder is less than that in the first term of the divisor, the process of division cannot be carried further, so that in such a case it terminates with a remainder.

6. Divide $4a^6 - 4x^6$ by $2a^2 - 2x^2$.

$$\begin{array}{l}
2a^2 - 2x^2)4a^6 - 4x^6(2a^4 + 2a^2x^2 + 2x^4 \\
\qquad 4a^6 - 4a^4x^2 \\
\hline
\qquad 4a^4x^2 - 4x^6 \\
\qquad 4a^4x^2 - 4a^2x^4 \\
\hline
\qquad 4a^2x^4 - 4x^6 \\
\qquad 4a^2x^4 - 4x^6 \\
\hline
\end{array}$$

87. By division it can easily be shewn that

$$\frac{x^n \pm a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + xa^{n-2} + a^{n-1} + \frac{R}{x - a},$$

and that

$$\frac{x^n \pm a^n}{x + a} = x^{n-1} - x^{n-2}a + x^{n-3}a^2 - \dots \mp xa^{n-2} \pm a^{n-1} \mp \frac{R}{x + a}.$$

The intermediate terms can easily be filled up when a particular value is given to n.

The form of the quotient being now known, let it be represented by Q, and the remainder, if any, by R, as above; it is required to find under what conditions there will be a remainder, and its value when there is one, without performing the division.

Writing Q for the quotient, the above expressions become

$$\frac{x^n \pm a^n}{x - a} = Q + \frac{R}{x - a},$$

and

$$\frac{x^n \pm a^n}{x + a} = Q + \frac{R}{x + a}.$$

Multiplying both sides of the first by $x - a$, and the second by $x + a$, they become respectively

$$x^n \pm a^n = (x - a)Q + R,$$

and

$$x^n \pm a^n = (x + a)Q + R.$$

Since the above are true for all values of x, put a for x in the first, and $-a$ for x in the second, and they become respectively

$$a^n \pm a^n = (a - a)Q + R,$$

and

$$(-a)^n \pm a^n = (\ -a + a)Q + R.$$

Again, since $(a - a)$ and $(-a + a)$ are each equal to 0, the first terms on the second side disappear; hence,

$$a^n \pm a^n = R,$$

and

$$(-a)^n \pm a^n = R.$$

Now in the first, whether n be odd or even, taking the *upper* sign, $R = 2a^n$; but taking the *under* sign, $R = 0$.

Again, in the second, if n be odd, $(-a)^n = -a^n$; but if n be even, $(-a)^n = +a^n$; hence, taking the upper sign and n odd, $R = 0$; but taking n even, $R = 2a^n$; if now in the second the under sign be taken and n odd, $R = -2a^n$, but if n be even, $R = 0$. From which the following theorems are derived:—

Theorem I. $x^n + a^n$ divided by $x - a$ leaves a remainder of $2a^n$, whether n be even or odd.

Theorem II. $x^n - a^n$ divided by $x - a$ leaves no remainder, whether n be even or odd.

Theorem III. $x^n + a^n$ divided by $x + a$ leaves no remainder if n be odd, but leaves a remainder of $2a^n$ if n be even.

Theorem IV. $x^n - a^n$ divided by $x + a$ leaves no remainder if n be even, but leaves a remainder of $-2a^n$ when n is odd.

EXERCISES.

1. Divide $4a^2 - 8ax + 4x^2$ by $2a - 2x$, . . $= 2a - 2x$.
2. ... $8a^4 - 8x^4$ by $2a^2 - 2x^2$, . . . $= 4a^2 + 4x^2$.
3. ... $a^3 + 5a^2x + 5ax^2 + x^3$ by $a^2 + 4ax + x^2$, $= a + x$.
4. ... $x^3 - 9x^2 + 27x - 27$ by $x - 3$, $= x^2 - 6x + 9$.
5. ... $x^4 - y^4$ by $x - y$, . . $= x^3 + x^2y + xy^2 + y^3$.
6. ... $a^3 - x^3$ by $a^2 + ax + x^2$, . . . $= a - x$.
7. ... $x^3 - 3x^2y + 3xy^2 - y^3$ by $x - y$, $= x^2 - 2xy + y^2$.

8. Divide $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ by $a^2 - 2ax + x^2$,
$= a^3 - 3a^2x + 3ax^2 - x^3$.

9. ... $6x^4 - 96$ by $3x - 6$, $= 2x^3 + 4x^2 + 8x + 16$.

10. ... $4a^6 - 25a^2x^4 + 20ax^5 - 4x^6$ by $2a^3 - 5ax^2 + 2x^3$,
$= 2a^3 + 5ax^2 - 2x^3$.

11. ... $6a^4 + 4a^3x - 9a^2x^2 - 3ax^3 + 2x^4$ by $2a^2 + 2ax - x^2$,
$= 3a^2 - ax - 2x^2$.

12. ... $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$ by $x^3 - 3x^2y + 3xy^2 - y^3$,
$= x^3 + 3x^2y + 3xy^2 + y^3$.

SYNTHETIC DIVISION.

I. When the powers of the letter or letters of the divisor and dividend *increase or decrease by any regular law*, the coefficients of the quotient may be obtained by observing that in performing the ordinary process of division, the coefficients of the terms of the divisor are multiplied by the coefficients of the quotient, and the products successively subtracted from the coefficients of the dividend; by which means, when the quantities are divisible without a remainder, there is at last nothing left.

If now the signs of all the terms of the divisor be changed from *plus* to *minus*, or from *minus* to *plus*, and the products in the corresponding columns added to the coefficients of the dividend, the sums will all be zeros; but if the coefficient of the first term of the divisor be omitted in performing the operation, the sums will be the coefficients of the quotient, if the coefficient of the first term of the divisor be one; and the coefficient of the quotient multiplied by the coefficient of the first term of the quotient when it is not one; in which case divide by the coefficient of the first term of the divisor, and the quotient will be the coefficient of the quotient.

The powers of the letters in the quotient always follow the same law as those of the dividend and divisor; and the exponent of the first term of the quotient is always the *difference* of the *exponents* of the *first* terms of the *dividend* and *divisor*, which being found, the others can easily be supplied.

In performing the work, the coefficients of like combinations of letters in the dividend, and the several products must be placed in the same vertical column, which will be effected by arranging the operation as in the following example.

1. Divide $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$ by $x^2 - 2x + 1$.

The coefficients of the divisor, with all the signs changed, except the first, form the first vertical column:—

$$\begin{array}{r|l} 1 & 1 - 6 + 15 - 20 + 15 - 6 + 1 = \text{coefs. of the dividend.} \\ +2 & \quad\ \ + 2 - \ 8 + 12 - \ 8 + 2 \\ -1 & \qquad\quad\ - \ 1 + \ 4 - \ 6 + 4 - 1 \\ \cline{2-2} & 1 - 4 + \ 6 - \ 4 + \ 1 \qquad\quad = \quad \ldots \quad \text{quotient.} \end{array}$$

Whence the quotient is $x^4 - 4x^3 + 6x^2 - 4x + 1$.

In the above operation, since the coefficient of the first term of the divisor is 1, the coefficient of the first term of the dividend is also the coefficient of the first term of the quotient, which is therefore 1. The second and third coefficients of the divisor are now multiplied by 1, and the products placed opposite to them and under the second and third coefficients of the dividend. The second vertical column is now added, which gives the second coefficient of the quotient $= -4$; the coefficients of the divisor are now multiplied by -4, and the product placed opposite to them and under the third and fourth coefficients of the dividend. The third vertical column is now added, which gives the third coefficient of the quotient, and so on till all the terms of the quotient are obtained.

If, after finding all the coefficients of the quotient, the sum of the coefficients or vertical columns to the right of the last coefficient of the quotient be not zeros, they are the coefficients of a remainder having the same literal parts as the terms of the dividend from which they arise; this remainder must be annexed to the quotient to give the true quotient.

2. Divide $x^6 - a^6$ by $x^3 + 2x^2a + 2xa^2 + a^3$.

Since, in this example the dividend wants 5 of the powers of x and a, their places must be supplied by ciphers; thus—

$$\begin{array}{r|l} 1 & 1 + 0 + 0 + 0 + 0 + 0 - 1 = \text{coefs. of the dividend.} \\ -2 & \quad - 2 + 4 - 4 + 2 \\ -2 & \qquad\quad - 2 + 4 - 4 + 2 \\ -1 & \qquad\qquad\quad - 1 + 2 - 2 + 1 \\ \cline{2-2} & 1 - 2 + 2 - 1 \qquad\qquad\quad = \quad \ldots \quad \text{quotient.} \end{array}$$

Hence $\quad x^3 - 2x^2a + 2xa^2 - a^3$. Answer.

3. Divide $24x^4 + 54x^3 + 33x^2 - 8x - 7$ by $6x^2 + 3x - 3$.

The coefficient of the first term of the divisor being 6, the sum of the coefficients in the several columns must be divided by 6 in

order to get the coefficient of the quotient. The work may be arranged thus—

$$\begin{array}{r|l}
6 & 24 + 54 + 33 - 8 - 7 = \text{coefs. of the dividend.} \\
-3 & \quad - 12 - 21 - 12 \\
+3 & \qquad\quad + 12 + 21 + 12 \\
6) & 24 + 42 + 24 \mid + 1 + 5 = \quad \ldots \quad \text{remainder.} \\
 & \quad 4 + 7 + 4 \qquad\qquad = \quad \ldots \quad \text{quotient.}
\end{array}$$

Hence the quotient is $= 4x^2 + 7x + 4 + \dfrac{x + 5}{6x^2 + 3x - 3}$.

II. The rule may be proved very simply as follows:—Let the line above the coefficients of the quotient be considered as the sign of equality, since the sum of the numbers in each vertical column above it is equal to the number below it; if now all the numbers above the line, except the coefficients of the dividend, be brought below it with their signs changed, retaining them in the same vertical columns, the equality will still exist; but the numbers below the line will then be the coefficients of the product arising from multiplying the coefficients of the quotient by those of the divisor; and since their product is equal to the coefficients of the dividend, the quotient, multiplied by the divisor, produces the dividend: hence the accuracy of the process.

The exercises in CASE IV. of Division may all be performed by this as well as by the common method, to which add the following

EXERCISES.

1. Divide $1 - 6x^5 + 5x^6$ by $1 - 2x + x^2$, $= 1 + 2x + 3x^2 + 4x^3 + 5x^4$
2. ... $x^6 - 140x^4 + 1050x^3 - 3101x^2 + 3990x - 1800$ by $x^3 - 12x^2 + 47x - 60$, $= x^3 + 12x^2 - 43x + 30$
3. ... $x^6 - 6x^5 + 20x^4 - 40x^3 + 50x^2 - 40x + 100$ by $x^3 - 2x^2 + 5x - 9$, $= x^3 - 4x^2 + 7x + 3 - \dfrac{15x^2 - 8x - 127}{x^3 - 2x^2 + 5x - 9}$
4. ... $x^6 - y^6$ by $x - y$, $= x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$

EXAMPLES WITH LITERAL EXPONENTS.

1. Divide $a^{2n} - x^{2n}$ by $a^n - x^n$.

$$\begin{array}{r}
a^n - x^n)\,a^{2n} - x^{2n}\,(a^n + x^n \\
a^{2n} - a^nx^n \qquad\quad \\
\hline
a^nx^n - x^{2n} \\
a^nx^n - x^{2n} \\
\hline
\end{array}$$

2. Divide $2a^{2n} - 4a^nx^n + 2x^{2n}$ by $a^n - x^n$.

$$\begin{array}{l}
a^n - x^n)2a^{2n} - 4a^nx^n + 2x^{2n}(2a^n - 2x^n \\
\qquad\quad 2a^{2n} - 2a^nx^n \\
\hline
\qquad\qquad - 2a^nx^n + 2x^{2n} \\
\qquad\qquad - 2a^nx^n + 2x^{2n} \\
\hline
\end{array}$$

3. ... $x^{3n} - 3x^{2n}y^n + 3x^ny^{2n} - y^{3n}$ by $x^n - y^n$.

$$\begin{array}{l}
x^n - y^n)x^{3n} - 3x^{2n}y^n + 3x^ny^{2n} - y^{3n}(x^{2n} - 2x^ny^n + y^{2n} \\
\qquad\quad x^{3n} - x^{2n}y^n \\
\hline
\qquad\qquad - 2x^{2n}y^n + 3x^ny^{2n} \\
\qquad\qquad - 2x^{2n}y^n + 2x^ny^{2n} \\
\hline
\qquad\qquad\qquad x^ny^{2n} - y^{3n} \\
\qquad\qquad\qquad x^ny^{2n} - y^{3n} \\
\hline
\end{array}$$

Any of the preceding exercises may be changed into exercises with literal exponents, by merely multiplying all the exponents in the divisor and dividend by n. The 3d example given here is the 7th of the preceding exercises, with all the exponents multiplied by n. The quotients will also be the same as in these exercises, with the exponents multiplied by n.

EXAMPLES WITH LITERAL COEFFICIENTS.

1. Divide $x^3 - (a + c)x^2 + (b + ac)x - bc$ by $x - c$.

$$\begin{array}{l}
x^3 - (a + c)x^2 + (b + ac)x - bc \left\{\begin{array}{l} x - c \\ x^2 - ax + b \end{array}\right. \\
x^3 - \qquad cx^2 \\
\hline
\qquad - ax^2 + (b + ac)x \\
\qquad - ax^2 + acx \\
\hline
\qquad\qquad bx - bc \\
\qquad\qquad bx - bc \\
\hline
\end{array}$$

2. Divide $x^3 - (a + c)x^2y + (b + ac)xy^2 - bcy^3$ by $x - cy$.

$$\begin{array}{l}
x^3 - (a + c)x^2y + (b + ac)xy^2 - bcy^3 \left\{\begin{array}{l} x - cy \\ x^2 - axy + by^2 \end{array}\right. \\
x^3 - \qquad cx^2y \\
\hline
\qquad - ax^2y + (b + ac)xy^2 \\
\qquad - ax^2y + acxy^2 \\
\hline
\qquad\qquad bxy^2 - bcy^3 \\
\qquad\qquad bxy^2 - bcy^3 \\
\hline
\end{array}$$

EXERCISES.

1. Divide $x^4 - (p + r)x^3 + (q + pr)x^2 - qrx$ by $x - r$,
$= x^3 - px^2 + qx$

2. ... $x^3 - (1 + a)x^2 + (a + b)x - b$ by $x - 1$,
$= x^2 - ax + b$

3. ... $x^5 - (1 + a)x^4 + (1 + a + b)x^3 - (a + b + c)x^2 +$
$(b+c)x-c$ by x^2-x+1, $= x^3 - ax^2 + bx - c$

4. ... $2abx^2 - (3bc - 2ad)xy - 3cdy^2$ by $bx + dy$,
$= 2ax - 3cy$

In the following and similar examples, the remainder is th same as the dividend, with r instead of x.

Divide $ax^3 + bx^2 + cx + d$ by $x - r$.

$$
\begin{array}{l}
ax^3 + bx^2 + cx + d \quad \{\ x - r \\
ax^3 - arx^2 \qquad\qquad\ \ ax^2 + (ar + b)x + (ar^2 + br + c) \\
\hline
\qquad (ar + b)x^2 + cx \\
\qquad (ar + b)x^2 - (ar^2 + br)x \\
\hline
\qquad\qquad (ar^2 + br + c)x + d \\
\qquad\qquad (ar^2 + br + c)x - (ar^3 + br^2 + cr) \\
\hline
\qquad\qquad\qquad ar^3 + br^2 + cr + d
\end{array}
$$

88. The same result will be obtained in dividing any simila polynomial by a binomial, in which the first term is the firs power of the quantity by reference to which the terms of th polynomial are arranged. This may be proved generally in th same manner as the properties in the last article were demon strated; for if $ax^n + bx^{n-1} + \ldots + mx + n$ be substituted fo $x^n \pm a^n$ and $x - r$, for $x - a$; by going over exactly the sam steps, we at last obtain, by substituting r for x (as x may have an value), $ar^n + br^{n-1} + \ldots + mr + n = R$.

EXERCISES.

Divide $ax^2 + bx + c$ by $x - r$.

... $ax^4 + bx^3 + cx^2 + dx + e$ by $x - r$.

The remainders will be, for the former example, $ar^2 + br + c$ and for the latter, $ar^4 + br^3 + cr^2 + dr + e$.

When the quantity, according to the powers of which any expression is arranged, has compound coefficients; as

$$ax^3 - (b + c)x^2 + (d + e)x + f,$$

it is sometimes convenient to write it thus—

$$ax^3 \begin{array}{r|} -b \\ -c \end{array} x^2 \begin{array}{r|} +d \\ +e \end{array} x + f.$$

EXAMPLE.

1. Divide $ax^3 + bx^2 + cx$ by $x - 1$.

$$\begin{array}{l}
x - 1)ax^3 + bx^2 + cx(ax^2 \begin{array}{r|} +a \\ +b \\ \end{array} x \begin{array}{r|} +a \\ +b \\ +c \end{array} \quad \text{quotient.} \\
\qquad\quad \underline{ax^3 - ax^2} \\
\qquad\qquad \begin{array}{r|} +a \\ +b \end{array} x^2 + cx \\
\qquad\qquad \underline{\begin{array}{r|} +a \\ +b \end{array} x^2 \begin{array}{r|} -a \\ -b \end{array} x} \\
\qquad\qquad\qquad\quad \begin{array}{r|} +a \\ +b \\ +c \end{array} x \\
\qquad\qquad\qquad\quad \underline{\begin{array}{r|} +a \\ +b \\ +c \end{array} x \begin{array}{r|} -a \\ -b \\ -c \end{array} \qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad a + b + c = \text{rem.}
\end{array}$$

EXERCISES.

1. Divide $ax^3 - bx^2 + cx - d$ by $x + 1$,
$= ax^2 - (a + b)x + (a + b + c)$, rem. $= -(a + b + c + d)$.
2. ... $x^2 - ax + b$ by $x - r$,
$= x + (r - a)$, rem. $= r^2 - ar + b$.
3. ... $1 - ax + bx^2 - cx^3$ by $1 + x$,
$= 1 - (1 + a)x + (1 + a + b)x^2 - (1 + a + b + c)x^3$,
rem. $= (1 + a + b + c)x^4$.

INFINITE SERIES.

89. An *infinite* series consists of an unlimited number of terms, which observe a certain law.

The *law* of a series is a relation existing between its terms, so that when some of them are known, the rest may be easily supplied.

Thus, in the infinite series $ax - a^2x^3 + a^3x^5 - a^4x^7 +$ &c. any term is found by multiplying the preceding by $- ax^2$.

90. A quotient may sometimes become an infinite series, arranged by the ascending powers of one of the letters.

EXAMPLES.

1. Divide 1 by $1 - x$.

$$
\begin{array}{llll}
1 - x)1 & & & (1 + x + x^2 + \&c. \\
\underline{1 - x} & & & \\
 & x & & \\
 & \underline{x - x^2} & & \\
 & & x^2 & \\
 & & \underline{x^2 - x^3} & \\
 & & & \underline{x^3 \ \&c.}
\end{array}
$$

It appears from this example that the quantity $\frac{1}{1-x}$ is equ

to an infinite series, or

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \&c.$$

After finding two or three terms of the series, the law of is generally perceptible, so that it may be extended any lengt without further division.

2. Divide $1 - ax + bx^2 - cx^3 + \&c.$ by $1 + x$.

$$
1 + x)1 - ax + bx^2 - cx^3 + \ldots \left(1 - \left.\begin{array}{r}1\\ a\end{array}\right| x + \left.\begin{array}{r}1\\ +a\\ +b\end{array}\right| x^2 - \&c.\right.
$$

$$
\begin{array}{l}
\underline{1 + x} \\
\left.\begin{array}{r}-1\\ -a\end{array}\right| x + bx^2 \\
\left.\begin{array}{r}-1\\ -a\end{array}\right| x \left.\begin{array}{r}-1\\ -a\end{array}\right| x^2 \\
\hline
\qquad \left.\begin{array}{r}+1\\ +a\\ +b\end{array}\right| x^2 - cx^3 \\
\qquad\qquad \underline{\&c. \ \&c.}
\end{array}
$$

EXERCISES.

1. Divide $1 + x$ by $1 - x$, . $= 1 + 2x + 2x^2 + 2x^3 + \&$
2. ... $1 - x$ by $1 + x$, . $= 1 - 2x + 2x^2 - 2x^3 + \&$
3. ... 1 by $1 + x$, . . $= 1 - x + x^2 - x^3 + \&$
4. ... $1 - ax^2 + bx^4 - cx^6 + \&c.$ by $1 - x^2$,

$= 1 + (1-a)x^2 + (1-a+b)x^4 + (1-a+b-c)x^6 + \&$

THE GREATEST COMMON MEASURE.

91. A *measure* of a quantity is any quantity that is contained in it exactly, or without a remainder. A measure of a quantity is also called an *aliquot part* or *submultiple* of the quantity.

Thus, 4 is a measure of 12, and 6 of 24. So 8 or 3 is a measure of 24. Also a^2 is a measure of a^6; a^2x is a measure of a^5x^2; and $2a^2z^3$ is a measure of $4a^3z^3$.

92. A quantity that is a measure of two or more quantities is called a *common measure* of these quantities.

Thus, 5 is a common measure of 15 and 20; 6 of 12 and 18: $3a^2$ is a common measure of $6a^4x^2$ and $12a^5x$; and $4x^2y^3$ is a common measure of $8a^2x^2y^4$, $12x^3y^3$, and $20x^3y^4z$.

93. The common measure of the highest dimension of two or more quantities is called their *greatest common measure.*

The greatest common measure of a^4x^3 and a^2x^5 is a^2x^3, and that of $2a^4x^2y$ and $6a^4y^2z$ is $2a^4y$.

94. A *simple measure* of a compound quantity is a measure of each of its terms.

Thus, a simple measure of $4a^2x^2 - 8a^3x^4 + 6a^2x^6$ is $2a^2x^2$, for this quantity is contained in each of the terms of the compound quantity.

95. Quantities that have a common measure are said to be *commensurable;* and those that have no common measure are called *incommensurable.*

96. Incommensurable quantities are also said to be *prime to each other*, or *relatively prime.*

97. A quantity that has no measure except itself and unity, is said to be *prime*, or *absolutely prime.*

98. A quantity that has a measure, or which is the product of two or more factors, is called a *composite quantity.*

Thus, ax, the product of the factors a and x, is a composite quantity. So is $4c^2y^3$ and $12a^5x^2$. So $x^2 - a^2$, which is the product of $x + a$ and $x - a$, is a composite quantity.

99. The preceding terms — prime, absolutely, and relatively; also composite, commensurable, and incommensurable—are also applied to numbers.

CASE I.—TO FIND THE GREATEST COMMON MEASURE OF QUANTITIES WHEN THEY ARE SIMPLE.

100. RULE. The greatest common measure of the literal parts of the quantities will contain the lowest power of each letter that is common to all the given quantities, and may be easily found by inspection.

If the quantities have numerical coefficients, their greatest common measure must be prefixed to that of the literal parts.

EXAMPLES.

1. Find the highest common measure of a^2x^2 and a^5x.

Both the quantities contain a and x, and their lowest powers are a^2 and x; hence a^2x is their greatest common measure.

2. Find the greatest common measure of $4a^3b^2x^5y^3$ and $6a^4x^2y^2$.

The greatest common measure of 4 and 6 is 2; therefore the greatest common measure of these quantities is $2a^3x^2y^2$.

3. Find the greatest common measure of $4a^4x^2z$, $8a^2z^2$, and $12a^3y^4z^2$.

The greatest common measure of 4, 8, and 12, is 4; hence the greatest common measure is $4a^2z$.

EXERCISES.

1. Find the greatest common measure of $x^2y^4z^3$ and x^4z^5, $\quad = x^2z^3$.
2. ... ... of $3a^4y^2$, $6a^5x^3y^5$, and $9a^6y^4z$, $\quad = 3a^4y^2$.
3. ... ... of $8ax^2y^4z^5$, $12x^5z^6$, and $24a^3x^3z^2$, $\quad = 4x^2z^2$.
4. ... ... of $6a^2xy^2$, $12a^3y^4z^5$, $9a^5x^3y^4$, and $24a^3y^6z$, $\quad = 3a^2y^2$.

CASE II.—WHEN THERE ARE TWO COMPOUND QUANTITIES.

101. Rule. Arrange the two quantities as a divisor and dividend; divide that which contains the highest power of the leading quantity by the other, and then divide the divisor by the remainder; then divide the last divisor by the last remainder; and so on, till there be no remainder; the last divisor is the greatest common measure.

It is indifferent which of the quantities is made the first dividend, when the highest power of the leading quantity is the same in both; but in other cases, the given quantity, which contains the highest power of the leading quantity, must be made the dividend.

If the remainder in any case does not contain the leading quantity—that is, if it is *independent* of that quantity—there is no common measure.

If there be a simple common measure of the two given quantities, it may at first be suppressed, in order to simplify the process; but as it will be a factor of the greatest common measure, the last divisor must be multiplied by it.

If any quantity be a simple measure of one of the quantities,

and not of the other, it must be expunged—that is, divided out (at least if it be in a divisor)—before dividing by it, as it can form no part of the common measure.

If the coefficient of the first term of any dividend be not divisible by the first term of the divisor, it may be made so by multiplying all the terms of this quantity by any quantity that will render it divisible.

EXAMPLES.

1. Find the greatest common measure of $x^3 - y^3$ and $x^5 - x^3y^2$.

The second quantity in this example contains x^3 in both its terms, and the first does not; expunging it, the second quantity becomes $x^2 - y^2$; and dividing the first by it,

$$\begin{array}{r} x^2 - y^2)x^3 - y^3(x \\ x^3 - xy^2 \\ \hline xy^2 - y^3 \\ \text{or } (x - y)y^2 \\ \hline \end{array}$$

As the remainder is not divisible by the divisor (86), it must be made the divisor after expunging y^2; then

$$\begin{array}{r} x - y)x^2 - y^2(x + y \\ x^2 - xy \\ \hline xy - y^2 \\ xy - y^2 \\ \hline \end{array}$$

Therefore $x - y$ is the greatest common measure of the two given quantities.

This might be more expeditiously effected by (69, Theo. III.) and (69, Theo. IV.), for $x^2 - y^2 = (x + y)(x - y)$ by (Theo. III.), and $x^3 - y^3 = (x^2 + xy + y^2)(x - y)$ by (Theo. IV.); therefore $x - y$ is evidently the greatest common measure.

2. Find the greatest common measure of $x^6 + a^3x^3$ and $x^4 - a^2x^2$.

The factor x^2 is common to both these quantities; it will therefore form a part of the common measure, or will be a factor of it. Expunging this factor, the results are $x^4 + a^3x$ and $x^2 - a^2$. The former quantity still contains a factor x common to both its terms, and the latter does not; it must therefore be expunged, which gives $x^3 + a^3$. Hence dividing

$$\begin{array}{r} x^2 - a^2)x^3 + a^3(x \\ x^3 - a^2x \\ \hline a^2x + a^3 \\ \text{or } (x + a)a^2 \end{array}$$

$$
\begin{array}{l}
x + a)x^2 - a^2(x - a \\
\quad\quad x^2 + ax \\
\hline
\quad\quad\quad - ax - a^2 \\
\quad\quad\quad - ax - a^2 \\
\hline
\end{array}
$$

Therefore the greatest common measure of $x^3 + a^3$ and $x^2 -$ or of $x^4 + a^3x$ and $x^2 - a^2$, is $x + a$; and that of the given qu tities is $x^2(x + a)$; or more concisely, $x^2 - a^2 = (x + a)(x -$ (69, Theo. III.), and $x^3 + a^3 = (x^2 - xa + a^2)(x + a)$ (69, Theo. V where $x + a$ is evidently the common measure.

3. Find the greatest common measure of $x^{13} - a^{13}$ and $x^5 - a$

$$
\begin{array}{l}
x^5 - a^5)x^{13} - a^{13}(x^8 + x^3a^5 \\
\quad\quad x^{13} - x^8a^5 \\
\hline
\quad\quad\quad x^8a^5 - a^{13} \\
\quad\quad\quad x^8a^5 - x^3a^{10} \\
\hline
\quad\quad\quad\quad x^3a^{10} - a^{13} \\
\quad\quad\quad\quad \text{or } (x^3 - a^3)a^{10}
\end{array}
$$

$$
\begin{array}{l}
x^3 - a^3)x^5 - a^5(x^2 \\
\quad\quad x^5 - x^2a^3 \\
\hline
\quad\quad\quad x^2a^3 - a^5 \\
\quad\quad\quad \text{or } (x^2 - a^2)a^3
\end{array}
$$

$$
\begin{array}{l}
x^2 - a^2)x^3 - a^3(x \\
\quad\quad x^3 - xa^2 \\
\hline
\quad\quad\quad xa^2 - a^3 \\
\quad\quad\quad \text{or } (x - a)a^2
\end{array}
$$

$$
\begin{array}{l}
x - a)x^2 - a^2(x + a \\
\quad\quad x^2 - xa \\
\hline
\quad\quad\quad xa - a^2 \\
\quad\quad\quad xa - a^2 \\
\hline
\end{array}
$$

The common measure is therefore $x - a$; but this might much more concisely shewn by article (87, Theo. II.)

4. Find the greatest common measure of $5a^5 + 10a^4x + 5a$ and $a^3x + 2a^2x^2 + 2ax^3 + x^4$.

$5a^3$ is a factor of the first only, and x of the second only; he suppressing these factors, and dividing, we have

$$
\begin{array}{l}
a^2 + 2ax + x^2)a^3 + 2a^2x + 2ax^2 + x^3(a \\
\quad\quad\quad\quad a^3 + 2a^2x + ax^2 \\
\hline
\quad\quad\quad\quad\quad\quad ax^2 + x^3 \\
\quad\quad\quad\quad\quad\quad \text{or } (a + x)x^2
\end{array}
$$

$$
\begin{array}{l}
a + x)a^2 + 2ax + x^2(a + x \\
\quad a^2 + ax \\
\hline
\quad\quad ax + x^2 \\
\quad\quad ax + x^2 \\
\hline
\end{array}
$$

The common measure therefore is $= a + x$.

5. Find the greatest common measure of $6a^4 - a^2x^2 - 12x^4$ and $9a^5 + 12a^3x^2 - 6a^2x^3 - 8x^5$.

$$
\begin{array}{l}
6a^4 - a^2x^2 - 12x^4)\ 9a^5 + 12a^3x^2 - 6a^2x^3 - 8x^5(3a \\
\quad 2 \\
\hline
18a^5 + 24a^3x^2 - 12a^2x^3 - 16x^5 \\
18a^5 - 3a^3x^2 - 36ax^4 \\
\hline
\quad 27a^3x^2 - 12a^2x^3 + 36ax^4 - 16x^5 \\
\text{or } (27a^3 - 12a^2x + 36ax^2 - 16x^3)x^2
\end{array}
$$

Rejecting the factor x^2, and making the former divisor the dividend, we have

$$
\begin{array}{l}
6a^4 - a^2x^2 - 12x^4 \left\{ \begin{array}{l} 27a^3 - 12a^2x + 36ax^2 - 16x^3 \\ 2a + 8x \end{array} \right. \\
9 \\
\hline
54a^4 - 9a^2x^2 - 108x^4 \\
54a^4 - 24a^3x + 72a^2x^2 - 32ax^3 \\
\hline
\quad 24a^3x - 81a^2x^2 + 32ax^3 - 108x^4 \\
\quad 9 \\
\hline
216a^3x - 729a^2x^2 + 288ax^3 - 972x^4 \\
216a^3x - 96a^2x^2 + 288ax^3 - 128x^4 \\
\hline
\quad - 633a^2x^2 - 844x^4 \\
\text{or } - 211x^2(3a^2 + 4x^2)
\end{array}
$$

$$
\begin{array}{l}
3a^2 + 4x^2)27a^3 - 12a^2x + 36ax^2 - 16x^3(9a - 4x \\
\quad 27a^3 + 36ax^2 \\
\hline
\quad\quad - 12a^2x - 16x^3 \\
\quad\quad - 12a^2x - 16x^3 \\
\hline
\end{array}
$$

Hence $3a^2 + 4x^2$ is the measure required.

In this example there are three instances in which it was necessary to multiply the dividend, in order that the coefficient of the first term should be divisible by that of the divisor—once by 2 and twice by 9.

The preceding rule for finding the greatest common measure
quantities depends on the following principles :—

102. *A quantity that measures two other quantities measures b*
their sum and their difference, and also any multiples (110) *of th*
quantities with their sum and difference.

Let M be a quantity, whether simple or compound, th
measures any two quantities A and B; and let m and n be t
quotients arising from dividing A and B by M. Then

$$A = mM,\ B = nM;$$

therefore, adding equal quantities to equals, it follows that

$$A + B = mM + nM = (m + n)M;$$

and therefore M is contained in $A + B$ as often as there are un
in the number, which is $= m + n$; that is, M measures the su
of A and B.

Again, taking the differences of equal quantities,

$$A - B = mM - nM = (m - n)M;$$

and therefore $A - B$ contains M exactly $(m - n)$ times; or M
a measure of $A - B$.

Also, if pA and qB are any multiples of A and B, it is evide
that $pA = mpM$, and $qB = nqM$, so that pA and qB are multipl
of M; and hence M measures also the sum and difference of p
and qB.

103. After performing the operations prescribed in the ru
the last divisor is a common measure of the given quantities, a
is also the greatest possible common measure.

For let A, B, be the given quantities; let A contain B, p tim
with a remainder C; and let B contain C, q times, with a remaind
M; and let C contain M, r times exactly. This process
represented below:

$$\begin{array}{l} B)A(p \\ \overline{C)B(q} \\ \overline{M)C(r} \\ \overline{\;0\;} \end{array}$$

Then, from the principles of division,

$$A = pB + C,\ B = qC + M,\ C = rM.$$

Since M is a measure of rM, it is so of its equal C, and ther
fore also of qC or of $qC + M$ (102); and hence also of B, a
therefore of pB or $pB + C$ (102), and therefore also of A. Hen
M is a measure of A and B. It is also the greatest comm
measure; for

$$C = A - pB,\ M = B - qC,$$

and any common measure of A and B, or of A and pB, must be
measure of $A - pB$ or C (102), and therefore also of qC; a

hence of $B - qC$ or M. That is, every measure of A and B is a measure of M; but M is the greatest measure of M, therefore M is the greatest common measure of A and B.

104. The quotients that result from dividing two quantities by their greatest common measure are relatively prime.

Let M be the greatest common measure of A and B, and let them contain M respectively m and n times, then m and n are relatively prime. For $A = mM$ and $B = nM$, and M being the greatest common measure of A and B, and therefore of mM and nM, m and n can have no common factor or measure; for if they had such a measure r, then rM would be the greatest common measure of A and B, which is contrary to hypothesis.

105. If two quantities have each a factor prime to the other quantity, and these factors be suppressed, the resulting quotients have their greatest common measure the same as the given quantities.

Let m and n be respectively factors of two quantities A and B, so that m shall be prime to B, and n to A; also, after suppressing these factors, let the quotients be A', B'; then the greatest common measure M of the two latter quantities is that of the given quantities. Let M be contained in A' and B' respectively, p and q times, then

$$A' = pM, B' = qM;$$

and therefore

$$A = mA' = mpM, B = nB' = nqM.$$

But m and n, being factors of A and B, and respectively prime to B and A, are relatively prime, and so are p and q; also m and q are so, since q is a factor of B, and also n and p for a similar reason; hence m and p being relatively prime to n and q, mp and nq are relatively prime (154); therefore M is the greatest common measure of A and B.

106. If a common factor of two given quantities be suppressed, the greatest common measure of the remaining quantities, multiplied by this factor, will be equal to that of the given quantities.

Let c be a common factor of A and B, and let $A = cA'$, $B = cB'$; then if M' be the greatest common measure of A', B', and M that of A, B, $M = cM'$. For let

$$A' = mM', \text{ and } B' = nM',$$
$$\text{then is } A = mcM', \text{ and } B = ncM';$$

and since m, n, are relatively prime (104), cM' is the greatest common measure of mcM' and ncM', or of A and B. That is, $M = cM'$.

107. If either of two given quantities be multiplied by a factor which is prime to the other, the greatest common measure of the product and the other given quantity is the same as that of the given quantities.

Let A, B, be the given quantities, and r a quantity prime to A,

the greatest common measure of A and rB is the same as that o
A and B.

For let M be the greatest common measure of A and B, so tha

$$A = mM,\ B = nM, \text{ and hence } rB = rnM.$$

Now, since m and n are prime relatively, and also m and r (for r i
prime to A), then m is prime to rn (152); and hence M is th
greatest common measure of mM and rnM, or of A and rB. Thi
theorem is a particular case of that in article (105), namely, whe
m in it is $= 1$.

EXERCISES.

1. Find the greatest common measure of $x^3 - a^2x$ and $x^3 - a^3$, $= x - a$
2. ... ... of $x^3 - c^2x$ and $x^2 + 2cx + c^2$ $= x + c$
3. ... ... of $a^2 - 5ax + 4x^2$ and $a^3 - a^2x + 3ax^2 - 3x^3$, $= a - x$
4. ... ... of $6a^3 - 6a^2x + 2ax^2 - 2x$ and $12a^2 - 15ax + 3x^2$, $= a - x$
5. ... ... of $a^2x^4 - a^2y^4$ and $x^5 + x^3y^2$ $= x^2 + y^2$
6. ... ... of $6a^4 - 5a^2x^2 - 6x^4$ and $4a^5 - 6a^3x^2 - 2a^2x^3 + 3x^5$, $= 2a^2 - 3x^2$
7. ... ... of $a^8 - x^8$ and $a^{19} - x^{19}$, $= a - x$
8. ... ... of $6x^5 - 4x^4 - 11x^3 - 3x$ $- 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$, $= 2x^3 - 4x^2 + x - 1$

CASE III.—TO FIND THE GREATEST COMMON MEASURE OF THREE QUANTITIES

108. RULE. Find first the greatest common measure of two o
them, and then that of this measure and the third quantity; thi
last measure will be the one required.

Let A, B, and C, be the three quantities, M the greates
common measure of A and B, and N that of M and C, then N i
also that of A, B, and C.

For any measure of A and B is one also of M, therefore any
measure of A, B, and C, must be a measure of M and C
likewise any measure of M being also one of A and B, any
measure of M and C is also a measure of A, B, and C. Henc
the greatest measure of M and C is also that of A, B, and C.

109. RULE. The greatest common measure of four quantitie
is found by first finding that of three of them, and then that o

this measure and the fourth quantity; this last measure is the one required. And the greatest common measure of five, or of any number of quantities, is found in a similar manner.

THE LEAST COMMON MULTIPLE OF QUANTITIES.

110. A *multiple* of a quantity is any quantity that contains it exactly.

Thus, 6 is a multiple of 2 or of 3, and 24 of 2, 3, 4, 6, 8, 12; $12a^2x^3$ is a multiple of $12a$, of $12ax^2$, of ax, &c. And $4(a-x)y^2$ is a multiple of $2(a-x)$, of $2y$, &c.

111. A quantity that contains two or more quantities is called a *common* multiple of them.

Thus, 12 is a multiple of 3 and 4, or it is a common multiple of these numbers. So $24ax^2$ is a common multiple of 12, $8ax$, $6x^2$, &c. And $8(x^2-y^2)x^2$ is a common multiple of $8(x^2-y^2)$, $4x^2$, &c.

CASE I.—TO FIND THE LEAST COMMON MULTIPLE OF TWO QUANTITIES WHEN THEY ARE SIMPLE.

112. RULE I. The least common multiple of the literal parts of any two simple quantities is the product of the highest powers of each of the letters contained in them, and may be found by inspection.

RULE II. The least common multiple of two quantities may also be found by dividing their product by their greatest common measure; or by dividing one of the quantities by their greatest common measure, and then multiplying the quotient by the other; and in the same manner may the least common multiple of two numbers be found.

If the quantities have numerical coefficients, their least common multiple must be prefixed to that of the literal parts.

EXAMPLES.

1. Find the least common multiple of $4a^2xy^3$ and $6x^4y^5$.

Let $M=$ the greatest common measure,

and $L=$ ... least ... multiple,

then $M=2xy^3$; therefore $L=\dfrac{4a^2xy^3\times 6x^4y^5}{2xy^3}$, or $=2a^2\times 6x^4y^5$

$=12a^2x^4y^5$.

But in this example the least common multiple may also be found by inspection.

Thus, the highest given powers of all the letters are a^2, x^4, and y^5, and the least common multiple of the literal part is therefore $a^2x^4y^5$.

2. Find the least common multiple of $6x^2y^3z^4$ and $8a^2x^5z^6$.

$$L = \frac{6 \times 8}{2} a^2x^5y^3z^6 = 24a^2x^5y^3z^6.$$

3. Find the least common multiple of $12a^5c^8x^6$ and $18a^3x^5$.

$$L = \frac{12 \times 18}{6} a^5c^8x^6 = 2 \times 18a^5c^8x^6 = 36a^5c^8x^6.$$

EXERCISES.

1. Find the least common multiple of $12a^4x^6$ and $18a^2x^2y^4$, and also of $10a^2c^4y^2$ and $15ac^2y^6z^2$, . = $36a^4x^6y^4$ and $30a^2c^4y^6z^2$.

2. Find the least common multiple of $18x^4y^5z$ and $24a^2x^2z^2$, and also of $25axy^4$ and $30a^4x^2z^3$, . . $= 72a^2x^4y^5z^2$ and $150a^4x^2y^4z^3$.

3. Find the least common multiple of $6ac^4z^6$ and $18c^2z^{10}$, and of $5a^4c^2x$ and $8a^3c^3z^4$, . . . = $18ac^4z^{10}$ and $40a^4c^3xz^4$.

4. Find the least common multiple of $120a^6x^4z^5$ and $64c^4xy^4z$, $= 960a^6c^4x^4y^4z^5$.

CASE II.—TO FIND THE LEAST COMMON MULTIPLE WHEN ONE OR BOTH OF THE GIVEN QUANTITIES ARE COMPOUND.

113. Rule. Divide the product of the two quantities by their greatest common measure; or, divide one of the quantities by the greatest common measure, and multiply the quotient by the other.

EXAMPLES.

1. Find the least common multiple of $4ax^2$ and $3(a - x)$.

Here evidently $M = 1$, therefore $L = 12ax^2(a - x)$.

2. Find the least common multiple of $6a^2y(a - x)$ and $4a(a^2 - x^2)$.

$M = 2a(a - x)$, therefore $L = 3ay \times 4a(a^2 - x^2) = 12a^2y \times (a^2 - x^2)$.

3. Find the least common multiple of $4a(a^2 - x^2)$ and $6x^2(a^3 - x^3)$.

$M = 2(a - x)$, therefore $L = 2a(a + x) \times 6x^2(a^3 - x^3) = 12ax^2(a + x)(a^3 - x^3) = 12ax^2(a^4 + a^3x - ax^3 - x^4)$.

114. The rule for finding the least common multiple of two quantities is easily derived thus :—

Let A and B be two quantities, and M their greatest common measure, and let

$$A = mM, \text{ and } B = nM,$$

then (104) m, n, are relatively prime, or have no common factor, and their least common multiple is mn (157); therefore the least common multiple of mM and nM is mnM, but

$$mnM = \frac{mnM^2}{M} = \frac{mM \cdot nM}{M} = \frac{A \cdot B}{M} = L.$$

EXERCISES.

1. Find the least common multiple of $6a^4x^2y$ and $8a^2(a+x)$, $= 24a^4x^2y(a+x)$.
2 ... ... of $8a^2(x^2-y^2)$ and $12ax^4$, $= 24a^2x^4(x^2-y^2)$.
3. ... ... of $4a^2(a^2-x^2)$ and $6ax^4 \times (a^4-x^4)$, $= 12a^2x^4(a^4-x^4)$.
4. ... ... of $6x^2(a-x)$ and $4xy(a-x)^2 \times (a+x)$, $= 12x^2y(a+x)(a-x)^2$.
5. ... ... of $8ax(a^3+x^3)$ and $12a^4 \times (a+x)^2$, $= 24a^4x(a+x)(a^3+x^3)$ or $24a^4x(a+x)^2(a^2-ax+x^2)$.
6. Find the least common multiple of $24a^6(a^3-x^3)$ and $5a^2x^2 \times (a+x)(a-x)$,
$= 120a^6x^2(a+x)(a^3-x^3)$ or $120a^6x^2(a+x)(a-x)(a^2+ax+x^2)$.

CASE III. — TO FIND THE LEAST COMMON MULTIPLE OF THREE OR MORE QUANTITIES.

115. Rule. Find the least common multiple of two of the quantities by the preceding rule, and then the least common multiple of this multiple and the third quantity; the result is the least common multiple of the three quantities.

If there be four quantities, find the least common multiple of three of them, and then the least common multiple of this multiple and the fourth quantity; the result is the least common multiple of the four quantities.

In a similar manner, the least common multiple of any number of quantities may be found.

If A, B, and C, be three quantities, and L the least common multiple of A and B, then the least common multiple of L and C will be that of A, B, and C. For every common multiple of A and

B is a multiple of L; therefore every common multiple of A, B and C is a multiple of L and C; also every multiple of L and C is a multiple of A, B, and C; consequently the least common multiple of L and C is the least common multiple of A, B, and C.

EXAMPLES.

1. Find the least common multiple of $4a^2$, $3a^3x$, and $6ax^2y^3$.

For the first two the least common multiple $= 12a^3x$; and for this quantity and the third, the least common multiple $= 12a^3x^2y^3$

2. Find the least common multiple of $8a^2(a+x)$, $3ax^2$, and $4x^2y$.

For the first two the least common multiple $= 24a^2x^2(a+x)$ for this and the third, the least common multiple $= 24a^2x^2y(a+x)$

3. Find the least common multiple of $3(a-x)$, $2a^4x^2(a-x)^2$, $3a^2$, and $6x^4$.

It is evident, by inspection, that the least common multiple $= 6a^4x^4(a-x)^2$.

4. Find the least common multiple of $6a^2x^4$, $3a(x-y)$, $12x^2(x^2-y^2)$, and $18a^2x(x^4-y^4)$.

It is evident that for the first three, the least common multiple $= 12a^2x^4(x^2-y^2)$; and for this quantity and the fourth, the least common multiple $= 36a^2x^4(x^4-y^4)$.

EXERCISES.

1. Find the least common multiple of $12a^2x^2$, $6a^3$, and $8x^4y^2$,

$= 24a^3x^4y^2$

2. ... ... of $8x^2(x-y)$, $3a^4x^2$, and $12axy^2$

$= 24a^4x^2y^2(x-y)$

3. ... ... of $10a^2x^2(x-y)$, $15x^5(x-y)^2$ and $12(x^2-y^2)$, $= 60a^2x^5(x-y)^2(x+y)$

4. Find the least common multiple of $8(a-x)$, $4x^2(a^3-x^3)$, $\times$ $6a^2(a^2-x^2)$,

$= 24a^2x^2(a^2-x^2)(a^2+ax+x^2)$ or $24a^2x^2(a^4+a^3x-ax^3-x^4)$

ALGEBRAIC FRACTIONS.

116. If a unit of anything be divided into any number of equal parts, one of these parts or any number of these parts is called a fraction.

Thus, if one inch, as the line AB, be divided into 4 equal parts, one of these parts, as Ac, is the fourth part of an inch, and is represented thus $\frac{1}{4}$; and three of these equal parts, as Ad, is called three-fourths of an inch, and is represented by $\frac{3}{4}$. In the algebraic fraction $\frac{1}{b}$, if $b = 4$, and if 1 denote an inch, then $\frac{1}{b}$ means the fourth of an inch. Likewise, if in the fraction $\frac{a}{b}$, $a = 3$ and $\frac{1}{b} = \frac{1}{4}$ of an inch, $\frac{a}{b}$ represents $\frac{3}{4}$ of an inch.

117. The greater the number of parts into which a unit is divided, the less is one of these parts in the same proportion.

Let one inch be divided into 4 equal parts, and another inch into 12 equal parts, then it is evident that 1 of the former parts will be equal to 3 of the latter, or $\frac{1}{4} = 3$ times $\frac{1}{12}$; that is, if the number of equal parts be made 3 times greater, each of the parts will be 3 times less. Thus $\frac{1}{a}$ is m times greater than $\frac{1}{ma}$; for if one unit be divided into a equal parts, and another into ma equal parts, one of the former parts will evidently be m times as great as one of the latter, whatever numbers a and m may represent.

118. Algebraic fractions are represented in the same manner as numerical ones, the upper quantity being called the *numerator*, and the lower the *denominator*, which together are also called the *terms* of the fraction.

119. An *improper* algebraic fraction is one whose numerator can be divided by the denominator, with or without a remainder.

Thus, $\frac{ax + b}{x}$, or $\frac{ax^2 - b}{a + x}$, are improper fractions. The numerator, however, may be either numerically greater or less than the denominator, when particular values are assigned to the letters.

120. The *reciprocal* of any quantity is unity divided by that quantity. Thus, $\frac{1}{a}$ is the reciprocal of a.

121. The denominator shews the number of equal parts into which the unit is divided, and the numerator shews the number of those parts that are taken.

Thus, in the fraction $\frac{3}{4}$, the 4 shews that 1 inch (fig. at 116) is divided into 4 equal parts, and the numerator 3 shews that three of these are taken; thus the denominator indicates the *denomination* or name of the part, and the numerator points out the *number* of those parts that are taken. In like manner, in the fraction $\frac{a}{b}$, b shews that a unit is divided into b parts, and a denotes that a parts of these are taken.

122. Any part of a unit, taken any number of times, is equal to the same part of that number of units.

Thus, $\frac{1}{4}$ of 1 inch, taken 3 times, or $\frac{3}{4}$ of an inch, is equal to $\frac{1}{4}$ of 3 inches. For 3 inches contains $\frac{1}{4}$ of one inch 12 times, or 3 inches = 12-fourths of an inch, or $= \frac{12}{4}$; and the fourth part of 12 fourths is just three-fourths, or $\frac{3}{4}$ of one inch. So $\frac{a}{b}$ is either the fraction $\frac{1}{b}$ of one unit taken a times, or it is the bth part of a units. If $a = 5$, and $b = 8$, then $\frac{1}{8}$ of one inch, taken 5 times, is $= \frac{1}{8}$ of 5 inches, or $= \frac{5}{8}$.

123. If the numerator of a fraction be multiplied by any number, the fraction is multiplied by the same number.

If in the fraction $\frac{3}{4}$, 3 be multiplied by 5, so is the fraction $\frac{3}{4}$ multiplied by 5, for $\frac{15}{4}$ is 5 times $\frac{3}{4}$; or $\frac{1}{4}$ of a unit, taken 15 times = 5 times $\frac{1}{4}$ or $\frac{5}{4}$ taken 3 times. So $\frac{20}{6} = \frac{5 \times 4}{6}$ is = 4 times $\frac{5}{6}$, or = 5 times $\frac{4}{6}$. So $\frac{3a}{b} = 3$ times $\frac{a}{b}$, and if m be any number, $\frac{ma}{b} = m$ times $\frac{a}{b}$.

124. If the denominator of a fraction be multiplied by any number, the fraction is divided by the same number.

If in the fraction $\frac{1}{4}$, the 4 be multiplied by 2, the fraction $\frac{1}{4}$ will be divided by 2, or $\frac{1}{8}$ is the half of $\frac{1}{4}$; for if 1 inch be divided into 8 equal parts, and another into 4 equal parts, any one of the latter parts is evidently equal to 2 of the former, or $\frac{1}{4} = 2$ times $\frac{1}{8}$. So $\frac{1}{2} = 3$ times $\frac{1}{6}$; $\frac{1}{4} = 3$ times $\frac{1}{12}$. So $\frac{1}{b}$ = twice $\frac{1}{2b}$, or = 3 times $\frac{1}{3b}$, or equal m times $\frac{1}{mb}$. But if $\frac{1}{4}$ = twice $\frac{1}{8}$, then $\frac{3}{4}$ = twice $\frac{3}{8}$; if $\frac{1}{2} = 3$ times $\frac{1}{6}$, then $\frac{3}{2} = 3$ times $\frac{3}{6}$. Also $\frac{3}{b}$ = twice $\frac{3}{2b}$, $\frac{a}{b}$ = twice $\frac{a}{2b}$, and generally $\frac{a}{b} = m$ times $\frac{a}{mb}$.

125. If the numerator of a fraction be divided by any number, the fraction is divided by the same number.

It has been shewn (123) that $\frac{15}{4}$ is $= 5$ times $\frac{3}{4}$; that $\frac{20}{6} =$ 5 times $\frac{4}{6}$, &c. Also $\frac{ma}{b}$ is m times $\frac{a}{b}$, so that the numerator ma being divided by m, gives a, and the fraction $\frac{a}{b}$ is an mth part of $\frac{ma}{b}$.

126. If the denominator of a fraction be divided by any number, the fraction is multiplied by the same number.

It has been shewn (124) that $\frac{1}{8}$ is the half of $\frac{1}{4}$, since $4 =$ one-half of 8; and that $\frac{1}{6}$ is the third of $\frac{1}{2}$, since $6 = 3$ times 2; also that $\frac{1}{b}$ is equal 3 times $\frac{1}{3b}$, or $\frac{1}{3b}$ = one-third of $\frac{1}{b}$; that $\frac{a}{2b}$ is = one-half of $\frac{a}{b}$; and generally that $\frac{a}{b}$ is m times $\frac{a}{mb}$.

127. If the terms of a fraction be both multiplied, or both divided, by the same quantity, its value is unchanged.

For if the numerator be multiplied by n, the fraction is also multiplied by n (123); and if then the denominator be multiplied by n, the fraction is again divided by n (124), and must therefore have its original value; thus $\frac{a}{b} = \frac{na}{nb}$. So $\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}$; $\frac{5}{8} = \frac{20}{32}$; and $\frac{24}{36} = \frac{2}{3}$.

CASE I.—TO REDUCE FRACTIONS TO THEIR LOWEST TERMS OR SIMPLEST FORM.

128. RULE. Divide the terms of the fraction by their greatest common measure.

EXAMPLES.

1. Reduce $\frac{4a^3x^2}{6a^4}$ to its simplest form.

The greatest common measure of the terms of the fraction is $2a^3$, and dividing them by it we have

$$\frac{4a^3x^2 \div 2a^3}{6a^4 \quad \div 2a^3} = \frac{2x^2}{3a}.$$

2. Reduce $-\frac{12x^2y^3z^4}{8x^2z^3}$ to its simplest form.

The greatest common measure $= 4x^2z^3$, and dividing the terms by it we have

$$\frac{-12x^2y^3z^4}{8x^2z^3} = -\frac{3y^3z}{2}.$$

3. Reduce $\frac{a(x^2 - a^2)}{x + a}$ to its lowest terms.

The greatest common measure is $= x + a$; hence $\frac{a(x^2 - a^2)}{x + a} =$ $a(x - a)$.

4. Reduce $\frac{6a^4x^2}{8a^2xy^4}$ to its lowest terms.

The greatest common measure $= 2a^2x$; hence $\frac{6a^4x^2}{8a^2xy^4} = \frac{3a^2x}{4y^4}$.

5. Reduce $\frac{x^4 - y^4}{x^6 - x^4y^2}$ to its simplest form.

The greatest common measure $= x^2 - y^2$; hence $\frac{x^4 - y^4}{x^6 - x^4y^2} =$ $\frac{x^2 + y^2}{x^4}$.

129. It is evident (127) that, after dividing the terms of the fraction by the greatest common measure, the resulting fraction will be of the same value as the given one.

EXERCISES.

1. Reduce $\frac{6a^2x^2}{4ax^3}$, and also $\frac{12x^5y^6z^3}{9a^2x^4z^5}$, to their simplest forms,

$$= \frac{3a}{2x} \text{ and } \frac{4xy^6}{3a^2z^2}.$$

2. ... $\frac{x^3 - xy^2}{x^4 - y^4}$ to its simplest form, . . $= \frac{x}{x^2 + y^2}$.

3. ... $\frac{x^3 - ax^2}{x^2 - 2ax + a^2}$ to its simplest form, . $= \frac{x^2}{x - a}$.

4. ... $\frac{a^3 + 2a^2x + ax^2}{a^3 + x^3}$ to its simplest form, $= \frac{a(a+x)}{a^2 - ax + x^2}$.

5. ... $\frac{a^3 - a^2x + 3ax^2 - 3x^3}{4a^2x - ax^2 - 3x^3}$ to its simplest form,

$$= \frac{a^2 + 3x^2}{x(4a + 3x)}.$$

6. ... $\frac{4a^4 - 4a^2x^2 + 4ax^3 - x^4}{6a^4 + 4a^3x - 9a^2x^2 - 3ax^3 + 2x^4}$ to its simplest form,

$$= \frac{2a^2 - 2ax + x^2}{3a^2 - ax - 2x^2}.$$

CASE II. — TO CONVERT A FRACTION INTO AN EQUIVALENT ONE, HAVING A DENOMINATOR EQUAL TO ANY MULTIPLE OF THE DENOMINATOR OF THE GIVEN FRACTION.

130. Rule. Multiply both terms of the fraction by such a quantity as will make the denominator equal to its given multiple.

The multiplier will be found by dividing the given multiple by the denominator.

When the given quantity is an integer, it may be first written in a fractional form by putting unity for its denominator; the multiplier will be the given multiple itself.

EXAMPLES.

1. Convert the fraction $\frac{5ax^2}{3y^4}$ to an equivalent one, having the denominator $12a^2y^6$.

$$\text{The multiplier is } = \frac{12a^2y^6}{3y^4} = 4a^2y^2;$$

$$\text{therefore } \frac{5ax^2}{3y^4} = \frac{5ax^2 \times 4a^2y^2}{3y^4 \times 4a^2y^2} = \frac{20a^3x^2y^2}{12a^2y^6}.$$

2. Convert $2a^4x^6$ to an equivalent fraction, having the denominator $3x^2y^4$.

$$2a^4x^6 = \frac{2a^4x^6}{1} = \frac{2a^4x^6 \times 3x^2y^4}{1 \times 3x^2y^4} = \frac{6a^4x^8y^4}{3x^2y^4}.$$

3. Convert the fraction $\frac{8x^4 - y^2}{9a^2xy^3}$ into an equivalent one, having the denominator $9a^3x^2y^3$.

$$\text{The multiplier } = \frac{9a^3x^2y^3}{9a^2xy^3} = ax;$$

$$\text{therefore } \frac{8x^4 - y^2}{9a^2xy^3} = \frac{(8x^4 - y^2)ax}{9a^2xy^3 \times ax} = \frac{8ax^5 - axy^2}{9a^3x^2y^3}.$$

4. Reduce $\frac{3a - 4x}{a^2 + x^2}$ to an equivalent fraction, having $2ax(a^4 - x^4)$ for its denominator.

The multiplier $= 2ax(a^2 - x^2)$, and the fraction

$$= \frac{(3a - 4x) \times 2ax(a^2 - x^2)}{(a^2 + x^2) \times 2ax(a^2 - x^2)} = \frac{2ax(a^2 - x^2)(3a - 4x)}{2ax(a^4 - x^4)}.$$

The rule for thus converting fractions depends on the principle stated in article (127).

EXERCISES.

1. Convert the fraction $\frac{3ax^2}{4}$ into an equivalent one, having $8a^2y^2$ for its denominator, ... $= \frac{6a^3x^2y^2}{8a^2y^2}$.

2. Convert $3a^4xy^2$ into an equivalent fraction, having $2a^2x^2$ for its denominator, ... $= \frac{6a^6x^3y^2}{2a^2x^2}$.

3. Convert $\frac{3a^2x^4}{7y^5}$ into an equivalent fraction, having $14x^3y^6z^2$ for its denominator, ... $= \frac{6a^2x^7yz^2}{14x^3y^6z^2}$.

4. Convert $\frac{3a^2 - 4x^2}{a + x}$ to an equivalent fraction, having $3x(a + x)$ for its denominator, ... $= \frac{9a^2x - 12x^3}{3x(a + x)}$.

5. Convert $\frac{a + x}{(a - x)^2}$ to an equivalent fraction, having $3x(a - x)^3$ for its denominator, ... $= \frac{3x(a^2 - x^2)}{3x(a - x)^3}$.

6. Convert $\frac{3(a^2 + x^2)}{a^2 - x^2}$ into an equivalent fraction, having $(a + x)(a^3 - x^3)$ for its denominator, $= \frac{3(a^2 + x^2)(a^2 + ax + x^2)}{(a + x)(a^3 - x^3)}$.

III.—TO REDUCE AN INTEGRAL OR MIXED QUANTITY TO A FRACTIONAL FORM.

131. Rule. Multiply the integral part by the denominator of the fractional part; to the product add the numerator with its proper sign, and place the sum as a numerator, under which write the denominator of the fractional part. An integral quantity is reduced to a fractional form, by making unity its denominator.

When the fractional part has the sign minus before it, the signs of all the terms of its numerator must be changed (59). Thus, if the fraction be $-\frac{a^2 - ax + x^2}{a^2 - x^2}$ it becomes $\frac{-a^2 + ax - x^2}{a^2 - x^2}$.

EXAMPLES.

1. Reduce $3a + \frac{x}{y}$ to a fractional form.

$$3a + \frac{x}{y} = \frac{3ay + x}{y}.$$

2. ... $6ax + \frac{ax^2 - a^3}{x}$ to a fractional form.

$$6ax + \frac{ax^2 - a^3}{x} = \frac{6ax^2 + ax^2 - a^3}{x} = \frac{7ax^2 - a^3}{x}.$$

3. ... $a^2 - ax + x^2 - \frac{a^3 - x^3}{a + x}$ to a fractional form.

The fraction $= \dfrac{(a^2-ax+x^2)(a+x)-a^3+x^3}{a+x} = \dfrac{a^3+x^3-a^3+x^3}{a+x} = \dfrac{2x^3}{a+x}$.

To prove the rule, let $a+\dfrac{b}{c}$ be given; then $a=\dfrac{a}{1}=\dfrac{ac}{c}$ (127), and $\dfrac{ac}{c}+\dfrac{b}{c}$, is = the sum of ac and b divided by c, or $=\dfrac{ac+b}{c}$. And thus it appears in a similar manner that $a-\dfrac{b}{c}=\dfrac{ac-b}{c}$.

EXERCISES.

1. Reduce $2+\dfrac{a}{x}$ to a fractional form, . $= \dfrac{2x+a}{x}$.
2. ... $3a+\dfrac{ax-a}{x}$ to a fractional form, $= \dfrac{4ax-a}{x}$.
3. ... $a+x+\dfrac{a^2+x^2}{a-x}$ to a fractional form, $= \dfrac{2a^2}{a-x}$.
4. ... $3a^2x-\dfrac{a^2x^2-a^3}{x}$ to a fractional form, $= \dfrac{2a^2x^2+a^3}{x}$.
5. ... $xy^2-\dfrac{x^2y^2-y^3}{x}$ to a fractional form, $= \dfrac{y^3}{x}$.
6. ... $a^2-x^2-\dfrac{a^4+x^4}{a^2+x^2}$ to a fractional form, $= -\dfrac{2x^4}{a^2+x^2}$.
7. ... $a-x+\dfrac{a^2+x^2-5}{a+x}$ to a fractional form, $= \dfrac{2a^2-5}{a+x}$.
8. ... $a^3-a^2x+ax^2-x^3-\dfrac{a^4+x^4-3}{a+x}$ to a fractional form, $= -\dfrac{2x^4-3}{a+x}$.

IV.—TO REDUCE AN IMPROPER FRACTION TO AN INTEGRAL OR MIXED QUANTITY.

132. Rule. Divide the numerator by the denominator, and the quotient will be the integral part; and to this annex the remainder with its proper sign as a numerator to the given denominator for the fractional part.

EXAMPLES.

1. Reduce $\frac{3ax + a^2}{x}$ to a mixed quantity.

$$\frac{3ax + a^2}{x} = 3a + \frac{a^2}{x}.$$

Here $3ax$ is divisible by x, and gives the quotient $3a$, and the remainder $= a^2$.

2. Reduce $\frac{2a^2x - x^3}{a}$ to a mixed quantity.

$$\frac{2a^2x - x^3}{a} = 2ax - \frac{x^3}{a}.$$

3. ... $\frac{5(a+x)x}{a+x}$ to an integral quantity.

$$\frac{5(a+x)x}{a+x} = 5x.$$

4. ... $\frac{a^2 - x^2 + 3}{a + x}$ to a mixed quantity.

The quotient is $= a - x$, and the remainder $= 3$; therefore

$$\frac{a^2 - x^2 + 3}{a + x} = a - x + \frac{3}{a + x}.$$

5. Reduce $\frac{a^3 + x^3 - 2a + x}{a + x}$ to a mixed quantity.

The quotient is $= a^2 - ax + x^2 - 2$, and the remainder $= 3x$; hence the fraction $= a^2 - ax + x^2 - 2 + \frac{3x}{a + x}$.

This rule is the converse of the preceding. It may be proved thus:—

Let the given fraction be $\frac{ax + b}{a}$, which is evidently $= \frac{ax}{a} + \frac{b}{a}$ or $= x + \frac{b}{a}$; for $\frac{ax}{a} = \frac{x}{1} = x$ (127).

EXERCISES.

1. Reduce $\frac{a^2 + x^2}{a}$ to a mixed quantity, . $= a + \frac{x^2}{a}$.

2. ... $\frac{5x - y}{x}$ to a mixed quantity, . . $= 5 - \frac{y}{x}$.

3. Reduce $\frac{a^2 - x^2 + 3c}{a + x}$ to a mixed quantity, $= a - x + \frac{3c}{a+x}$

4. ... $\frac{a^4 - x^4}{a^2 + x^2}$ to an integral quantity, $= a^2 - x^2$

5. ... $\frac{a^3 + x^3 - 2a^2 + x^2}{a + x}$ to a mixed quantity,

$$= a^2 - ax + x^2 - 2a + 2x - \frac{x^2}{a + x}$$

6. ... $\frac{a^2 - 2ax + x^2 - x^2}{a - x}$ to a mixed quantity, $= a - x - \frac{x^2}{a - x}$

7. ... $\frac{4ax - 2x^2 - a^2}{2a - x}$ to a mixed quantity, $= 2x - \frac{a^2}{2a - x}$

8. ... $\frac{12x^3 - 3x^2}{4x^3 - x^2 - 4x + 1}$ to a mixed quantity, $= 3 + \frac{3}{x^2 - 1}$

V.—TO REDUCE FRACTIONS HAVING DIFFERENT DENOMINATORS TO EQUIVALENT FRACTIONS HAVING A COMMON DENOMINATOR.

133. When two or more of the denominators have a common measure, find the least common multiple of these denominators and reduce all the fractions to equivalent ones, having this multiple as their denominator (130).

134. When the denominators are prime to each other, multiply each numerator by all the denominators except its own, for the new numerators, and all the denominators together, for a common denominator.

EXAMPLES.

1. Reduce $\frac{3a}{4x^2}$ and $\frac{6ax}{8xy^2}$ to their equivalents, with a common denominator.

The least common multiple of the denominators is $= 8x^2y^2$; hence the fractions become

$$\frac{3a \times 2y^2}{4x^2 \times 2y^2} \text{ and } \frac{6ax \times x}{8xy^2 \times x}, \text{ or } \frac{6ay^2}{8x^2y^2} \text{ and } \frac{6ax^2}{8x^2y^2}.$$

2. Reduce $\frac{2a}{5x}$ and $\frac{3x}{8a}$ to their equivalents, with a common denominator.

Here the denominators are relatively prime; hence, by the second rule, the fractions become

$$\frac{2a \times 8a}{5x \times 8a}, \text{ and } \frac{3x \times 5x}{8a \times 5x}; \text{ or } \frac{16a^2}{40ax}, \text{ and } \frac{15x^2}{40ax}.$$

3. Reduce $\frac{6a}{7y}$, $\frac{3ay}{4x}$, and $\frac{2x}{3a}$, to their equivalents, with a common denominator.

By the second rule, these fractions become

$$\frac{6a \times 4x \times 3a}{7y \times 4x \times 3a}, \frac{3ay \times 7y \times 3a}{4x \times 7y \times 3a}, \text{ and } \frac{2x \times 7y \times 4x}{3a \times 7y \times 4x};$$

$$\text{or } \frac{72a^2x}{84axy}, \frac{63a^2y^2}{84axy}, \text{ and } \frac{56x^2y}{84axy}.$$

4. Reduce $\frac{4ax}{15(a^2 - x^2)}$, $\frac{3y}{(a + x)x}$, and $\frac{2ay}{9x}$, to their equivalents, with a common denominator.

The least common multiple of the denominators $= 45x(a^2-x^2)$; hence the fractions become

$$\frac{4ax \times 3x}{15(a^2 - x^2) \times 3x}, \frac{3y \times 45(a - x)}{(a + x)x \times 45(a - x)}, \text{ and } \frac{2ay \times 5(a^2 - x^2)}{9x \times 5(a^2 - x^2)};$$

$$\text{or } \frac{12ax^2}{45x(a^2 - x^2)}, \frac{135(a - x)y}{45x(a^2 - x^2)}, \text{ and } \frac{10ay(a^2 - x^2)}{45x(a^2 - x^2)}.$$

The principle on which the rule for this operation is founded is that stated in article (127).

EXERCISES.

Reduce the following fractions to their equivalents with common denominators :—

1. $\frac{3}{4x}$ and $\frac{2x}{3a}$, ... $= \frac{9a}{12ax}$ and $\frac{8x^2}{12ax}$.

2. $\frac{5xy}{8a}$ and $\frac{3y}{4a^2x^2}$, ... $= \frac{5ax^3y}{8a^2x^2}$ and $\frac{6y}{8a^2x^2}$.

3. $\frac{5a}{3(a^2 - x^2)}$ $\frac{9x}{4(a - x)}$, $= \frac{20a}{12(a^2 - x^2)}$ and $\frac{27x(a + x)}{12(a^2 - x^2)}$.

4. $\frac{a}{3x}, \frac{3x}{4a}$, and $\frac{6(a-x)}{15(a+x)}$,

$$= \frac{20a^2(a+x)}{60ax(a+x)}, \frac{45x^2(a+x)}{60ax(a+x)}, \text{ and } \frac{24ax(a-x)}{60ax(a+x)}.$$

5. $\frac{3ax}{a+x}, \frac{2a}{3(a-x)}$, and $\frac{3}{4(a^2-x^2)}$,

$$= \frac{36ax(a-x)}{12(a^2-x^2)}, \frac{8a(a+x)}{12(a^2-x^2)}, \text{ and } \frac{9}{12(a^2-x^2)}.$$

6. $\frac{a^2+x^2}{a^2+ax+x^2}, \frac{3x^2}{a^3-x^3}$, and $\frac{2a^2}{a-x}$,

$$= \frac{(a-x)(a^2+x^2)}{a^3-x^3}, \frac{3x^2}{a^3-x^3}, \text{ and } \frac{2a^2(a^2+ax+x^2)}{a^3-x^3}.$$

7. $\frac{x+y}{x-y}, \frac{x-y}{x+y}$, and $\frac{x^2+y^2}{x^2-y^2}$, . . $= \frac{(x+y)^2}{x^2-y^2}, \frac{(x-y)^2}{x^2-y^2}$, and $\frac{x^2+y^2}{x^2-y^2}$.

8. $\frac{6x}{5y}, \frac{3(a-x)}{x+y}, \frac{2(a+x)}{3(x-y)}$, and $\frac{5}{3(x^2-y^2)x}$,

$$= \frac{18x^2(x^2-y^2)}{15xy(x^2-y^2)}, \frac{45xy(a-x)(x-y)}{15xy(x^2-y^2)}, \frac{10xy(a+x)(x+y)}{15xy(x^2-y^2)}, \frac{25y}{15xy(x^2-y^2)}.$$

VI.—ADDITION OF ALGEBRAIC FRACTIONS.

135. Rule. To add algebraic fractions, first reduce them to a common denominator (133 or 134); then add the numerators; and their sum being placed as a numerator to the common denominator, will give the required sum.

If there are any integral quantities, prefix their sum to that of the fractions.

EXAMPLES.

1. Add together $\frac{3}{4}, \frac{5}{6a}$, and $\frac{8}{3x}$.

$$\text{The sum} = \frac{9ax + 10x + 32a}{12ax}.$$

In this example we find that $12ax$ is the common denominator for the three fractions; the terms of the first are multiplied by $3ax$, those of the second by $2x$, and those of the third by $4a$.

2. Add together $\frac{2a}{3x}$, $\frac{3x}{4a}$, and $\frac{a^2+x^2}{6ax}$.

$$\text{Sum} = \frac{8a^2 + 9x^2 + 2a^2 + 2x^2}{12ax} = \frac{10a^2 + 11x^2}{12ax}.$$

Here the terms of the first fraction are multiplied by $4a$, those of the second by $3x$, and those of the third by 2.

3. Add together $3a - 2x + \frac{a+x}{a-x}$, $3x - 2a - \frac{a-x}{a+x}$, $2a - \frac{a}{x}$, and $3x$.

$$\text{The sum} = 3a + 4x + \frac{(a+x)^2x - (a-x)^2x - a(a^2-x^2)}{x(a^2-x^2)} = 3a + 4x +$$

$$\frac{a^2x + 2ax^2 + x^3 - a^2x + 2ax^2 - x^3 - a^3 + ax^2}{x(a^2-x^2)} = 3a + 4x + \frac{5ax^2 - a^3}{x(a^2-x^2)}.$$

The rule for addition of fractions depends on the principle stated in article (127), and on this axiom, that $\frac{3}{4} + \frac{2}{4} = \frac{5}{4}$; or $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$.

EXERCISES.

1. Add $\frac{3}{8}$, $\frac{2}{5x}$, $\frac{a}{x}$, and $\frac{1}{x^2}$, . . $= \frac{15x^2 + 16x + 40ax + 40}{40x^2}$.

2. ... $\frac{5+x}{y}$, $\frac{3-ax}{ay}$, and $\frac{b}{3a}$, . . . $= \frac{15a + by + 9}{3ay}$.

3. ... $3 + \frac{2a}{x}$, $5 - \frac{3a-2x}{x}$, and $7 + \frac{x-a}{a}$,

$$= 15 - \frac{a^2 - ax + x^2}{ax}, \text{ or } 16 + \frac{x^2 - a^2}{ax}.$$

4. ... $\frac{a-x}{ax}$, $\frac{a+x}{x^2}$, $\frac{x}{a^2}$, and $\frac{a^2-x^2}{4ax^2}$, $= \frac{8a^2x - 5ax^2 + 5a^3 + 4x^3}{4a^2x^2}$.

5. ... $3a + \frac{a+x}{ax}$, $2a - \frac{a-x}{3x^2}$, and $\frac{a+x}{4a^2}$,

$$= 5a + \frac{3x^3 + 15ax^2 + 16a^2x - 4a^3}{12a^2x^2}.$$

6. ... $a - x + \frac{a^2+x^2}{a+}$ [illegible] $- \frac{a^2-ax}{a+x}$, $2x - \frac{3a^2-2x^2}{a-x}$, and $-4a - \frac{a^3+x^3}{a^2-x^2}$, ... $= x - \frac{2(2a^3 + a^2x - ax^2)}{a^2-x^2}$.

VII.—SUBTRACTION OF ALGEBRAIC FRACTIONS.

136. Rule. To subtract algebraic fractions, reduce them first to a common denominator, then take the difference between the numerators, and write it over the denominator.

When there are integral quantities, find their difference, and prefix it to that of the fractions.

EXAMPLES.

1. From $\frac{3a}{4}$ subtract $\frac{8x}{5}$.

$$\text{Difference} = \frac{15a - 32x}{20}.$$

In this example the terms of the first fraction are multiplied by 5, and those of the second by 4 (134).

2. From $\frac{3a^2}{2x}$ subtract $\frac{5x^2}{3a}$.

$$\text{Difference} = \frac{9a^3 - 10x^3}{6ax}.$$

3. From $2a - \frac{ax}{3}$ subtract $a + \frac{2ax}{5}$.

$$\text{Difference} = 2a - a - \frac{5ax + 6ax}{15} = a - \frac{11ax}{15}.$$

The terms of the first fraction in this example are multiplied by 5, and those of the second by 3; and as the latter is to be subtracted, its sign is changed, the prefixed sign $(-)$ referring to the two terms $5ax$ and $6ax$.

4. From $3a - 2x - \frac{ax - x^2}{x^2 - 1}$ subtract $2a - x - \frac{a - x}{x + 1}$.

$$\text{Difference} = 3a - 2x - \frac{ax - x^2}{x^2 - 1} - \left(2a - x - \frac{a - x}{x + 1}\right)$$

$$= 3a - 2x - \frac{ax - x^2}{x^2 - 1} - 2a + x + \frac{a - x}{x + 1}$$

$$= a - x + \frac{(a - x)(x - 1)}{x^2 - 1} - \frac{ax - x^2}{x^2 - 1}$$

$$= a - x + \frac{ax - a - x^2 + x - ax + x^2}{x^2 - 1}$$

$$= a - x + \frac{x - a}{x^2 - 1}.$$

In this example the quantity to be subtracted is enclosed between parentheses, and the sign (−) prefixed, as it is to be subtracted; the parentheses are then removed, and the signs of all the terms of this quantity are changed according to the rule for subtraction (59).

The rule depends on the principle in article (127); and on this axiom that $\frac{3}{4} - \frac{2}{4}$ is $= \frac{1}{4}$; or $\frac{7}{8} - \frac{4}{8} = \frac{7-4}{8} = \frac{3}{8}$; that is, the difference between 7-eighths of unity and 4-eighths is = 3-eighths; and generally $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$.

EXERCISES.

1. From $\frac{2ax}{3}$ subtract $\frac{5ax}{2}$, ... $= -\frac{11ax}{6}$.

2 ... $\frac{3}{4a}$ subtract $\frac{5}{2x}$, ... $= \frac{3x - 10a}{4ax}$.

3. ... $2a + \frac{3}{4}$ subtract $a + \frac{1}{8}$, . . . $= a + \frac{5}{8}$.

4. ... $16 + \frac{3a}{4x}$ subtract $7 + \frac{4x}{3a}$, . $= 9 + \frac{9a^2 - 16x^2}{12ax}$.

5. ... $2a - 3x + \frac{a-x}{a}$ subtract $a - 5x + \frac{x-a}{x}$, $= a + 2x + \frac{a^2 - x^2}{ax}$.

6. ... $3a^2 - 4x^2 + \frac{x-y}{x+y}$ subtract $2a^2 - x^2 + \frac{x+y}{x-y}$,

$$= a^2 - 3x^2 - \frac{4xy}{x^2 - y^2}.$$

7. ... $5ax - \frac{ax - a^2}{x+y}$ subtract $3ax + \frac{a^2 + ax}{x-y}$,

$$= 2ax - \frac{2a(x^2 + ay)}{x^2 - y^2}.$$

8. ... $a + x + \frac{x}{x^2-y^2}$ subtract $a - x + \frac{1}{x+y}$, $= 2x + \frac{y}{x^2 - y^2}$.

9. ... $3y + \frac{y}{a}$ subtract $y - \frac{y-b}{c}$, $= 2y + \frac{(a+c)y - ab}{ac}$.

10. ... $\frac{3x^2 - 4ax + a^2}{a^2 - x^2}$ subtract $\frac{a-x}{a+x}$, . $= -\frac{2x}{a+x}$.

VIII.—MULTIPLICATION OF ALGEBRAIC FRACTIONS.

137. Rule I. Multiply the numerators together, for the numerator of the product, and multiply the denominators together, for the denominator of the product.

138. If the numerator of any of the given fractions, and also a denominator of any of them, have a common factor, it may be cancelled, and the quotients taken as new terms.

If any of the given quantities is a mixed quantity, it must first be reduced to a fractional form (131).

EXAMPLES.

1. Multiply $\frac{3a}{4}$ by $\frac{5x}{8}$.

$$\frac{3a}{4} \times \frac{5x}{8} = \frac{15ax}{32}.$$

2. ... $\frac{3xy}{7}, \frac{2x}{5}$, and $\frac{3y}{11}$.

$$\frac{3xy}{7} \times \frac{2x}{5} \times \frac{3y}{11} = \frac{18x^2y^2}{385}.$$

3. ... $\frac{12a}{5x}$ by $\frac{8x}{15a}$.

$$\frac{12a}{5x} \times \frac{8x}{15a} = \frac{96ax}{75ax} = \frac{32}{25}.$$

But the same result may be found by cancelling the common factor $3a$ from $12a$ and $15a$, and the factor x from $8x$ and $5x$; thus—

$$\frac{12a}{5x} \times \frac{8x}{15a} = \frac{4}{5} \times \frac{8}{5} = \frac{32}{25};$$

or thus,
$$\frac{\overset{4}{12a}}{\underset{5}{5x}} \times \frac{\overset{8}{8x}}{\underset{5}{15a}} = \frac{32}{25}.$$

The common factors of the terms of the given fractions are generally more easily found than that of their products, and ought therefore to be cancelled.

4. Multiply $\frac{3a}{4x}$, $\frac{16x}{9}$, and $\frac{8}{15a}$.

$$\frac{3a}{4x} \times \frac{16x}{9} \times \frac{8}{15a} = \frac{1}{1} \times \frac{4}{9} \times \frac{8}{5} = \frac{32}{45}.$$

Here $3a$ and $15a$ have the common factor $3a$, and dividing by it, the quotients are 1 and 5, which are put in place of $3a$ and $15a$. So $16x$ and $4x$ have $4x$ as a common factor.

5. Multiply $\frac{(a-x)x}{a}$ and $\frac{ax}{a^2 - x^2}$.

$$\frac{(a-x)x}{a} \times \frac{ax}{a^2 - x^2} = \frac{x}{1} \times \frac{x}{a+x} = \frac{x^2}{a+x}.$$

Here $(a-x)x$ and $a^2 - x^2$ have the common factor $a - x$, by cancelling which the quotients become x, and $a + x$; also, ax and a have a as a common factor.

6. Multiply $a - \frac{a^2 + x^2}{a}$ by $x + \frac{a^2 - x^2}{x}$.

$$\left(a - \frac{a^2+x^2}{a}\right)\left(x + \frac{a^2-x^2}{x}\right) = \frac{a^2-a^2-x^2}{a} \cdot \frac{x^2+a^2-x^2}{x} = \frac{-x^2}{a} \cdot \frac{a^2}{x} = -ax.$$

The rule may be proved thus:—If $\frac{a}{b}$ is to be multiplied by 1, the product is $\frac{a}{b}$; again, if it is to be multiplied by the dth part of 1, or by $\frac{1}{d}$, the product must be d times less than $\frac{a}{b}$; that is, $\frac{a}{bd}$ (124); and if $\frac{a}{b}$ is to be multiplied by $\frac{c}{d}$, the product must be c times greater than before, since $\frac{c}{d}$ is c times $\frac{1}{d}$; therefore the product must be $\frac{a}{bd}$ taken c times; that is, $\frac{ac}{bd}$ (123).

139. Rule II. To multiply a fraction by an integral quantity, either multiply the numerator or divide the denominator by it.

This rule is evident from articles (123) and (126).

Thus, $\frac{5a}{12x} \times 4x = \frac{5a}{12x} \times \frac{4x}{1} = \frac{20ax}{12x} = \frac{5a}{3}$, or $\frac{5a}{12x} \times 4x = \frac{5a}{3}$.

EXERCISES.

1. Multiply $\frac{2a}{3}$ and $\frac{4a}{5}$, ... $= \frac{8a^2}{15}$.

2. ... $\frac{3x^2}{10y}$ by $5y$, ... $= \frac{3x^2}{2}$.

3. ... $\frac{6a}{2}$, $\frac{5x}{3}$, and $\frac{8}{5ax}$, ... $= 8$.

4. ... $\frac{3(a+x)}{2}$ and $\frac{4x}{a+x}$, ... $= 6x$.

5. ... $\frac{a-x}{x^2}$ by $\frac{a^2}{a^2-x^2}$, ... $= \frac{a^2}{x^2(a+x)}$.

6. ... $\frac{x}{a+x}$, $\frac{a^2-x^2}{x^2}$, and $\frac{a}{a-x}$, . . . $= \frac{a}{x}$.

7. ... $\frac{x^2+y^2}{x-y}$, $\frac{x^2-y^2}{x+y}$, and $\frac{3a}{x}$, . . $= \frac{3a(x^2+y^2)}{x}$.

8. ... $\frac{a+x}{6a}$, $\frac{4a^2x^2}{a-x}$, and $\frac{a^2-x^2}{a^3+x^3}$, $= \frac{2ax^2(a+x)}{3(a^2-ax+x^2)}$.

9. ... $2a - \frac{a^2-x^2}{a}$ and $3x + \frac{a^2+x^2}{x}$, $= \frac{a^4+5a^2x^2+x^4}{ax}$.

IX.—DIVISION OF ALGEBRAIC FRACTIONS.

140. RULE I. Invert the divisor, and proceed as in Multiplication.

When mixed quantities are given, first reduce them to fractional forms.

EXAMPLES.

1. Divide $\frac{6ax}{5}$ by $\frac{4x}{3}$.

$$\frac{6ax}{5} \div \frac{4x}{3} = \frac{6ax}{5} \times \frac{3}{4x} = \frac{3a}{5} \times \frac{3}{2} = \frac{9a}{10}.$$

2. ... $\frac{4x^2}{9a^2}$ by $\frac{3a^3}{10x^3}$.

$$\frac{4x^2}{9a^2} \div \frac{3a^3}{10x^3} = \frac{4x^2}{9a^2} \times \frac{10x^3}{3a^3} = \frac{40x^5}{27a^5}.$$

3. Divide $\frac{8(a-x)}{x^2}$ by $\frac{5(a^2-x^2)}{3x}$.

$$\frac{8(a-x)}{x^2} \div \frac{5(a^2-x^2)}{3x} = \frac{8(a-x)}{x^2} \times \frac{3x}{5(a^2-x^2)}$$

$$= \frac{8}{x} \times \frac{3}{5(a+x)} = \frac{24}{5x(a+x)}.$$

4. ... $3a - \frac{a^2+x^2}{4a}$ by $5x - \frac{x^2+a^2}{3x}$.

$$\left(3a - \frac{a^2+x^2}{4a}\right) \div \left(5x - \frac{x^2+a^2}{3x}\right) = \frac{11a^2-x^2}{4a} \times \frac{3x}{14x^2-a^2}$$

$$= \frac{3x(11a^2-x^2)}{4a(14x^2-a^2)}.$$

The rule may thus be proved:—Let $\frac{a}{b}$ be divided by 1, the quotient is evidently $\frac{a}{b}$; and if it be divided by the dth part of 1, the quotient will be d times greater than before, or $\frac{ad}{b}$ (123); again, if $\frac{a}{b}$ be divided by c times $\frac{1}{d}$ or $\frac{c}{d}$, the quotient will be c times less than the former quotient $\frac{ad}{b}$, or it will be $\frac{ad}{bc}$ (124). But $\frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}$, and hence $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

141. Rule II. To divide a fraction by an integral quantity, either multiply the denominator or divide the numerator by it.

This is evident from articles (124) and (125).

Thus $\frac{12x}{5} \div 4x = \frac{12x}{5} \times \frac{1}{4x} = \frac{12x}{20x} = \frac{3}{5}$, or $\frac{12x}{5} \div 4x = \frac{3}{5}$.

EXERCISES.

1. Divide $\frac{3x}{2}$ by $\frac{5a}{7}$, ... $= \frac{21x}{10a}$.

2. ... $\frac{8ax}{5}$ by $\frac{15x}{2}$, ... $= \frac{16a}{75}$.

3. Divide $\frac{3a^2x}{7}$ by $\frac{3ax^2}{14}$, ... $= \frac{2a}{x}$.

4. ... $\frac{15x}{3}$ by $4ax$, ... $= \frac{5}{4a}$.

5. ... $\frac{16ax}{5}$ by $\frac{4x}{15}$, ... $= 12a$.

6. ... $\frac{3(a^2 - x^2)}{x}$ by $\frac{2(a + x)}{a - x}$, . . $= \frac{3(a - x)^2}{2x}$.

7. ... $\frac{32(a^3 - x^3)}{a^2 + x^2}$ by $\frac{8(a - x)}{a + x}$, $= \frac{4(a + x)(a^2 + ax + x^2)}{a^2 + x^2}$.

8. ... $2a + x - \frac{a^2 + x^2}{x}$ by $3y - \frac{b^2 + y^2}{y}$, $= \frac{y(2ax - a^2)}{x(2y^2 - b^2)}$.

9. ... $\frac{2x^2}{a^3 + x^3}$ by $\frac{x}{a + x}$, . . . $= \frac{2x}{a^2 - ax + x^2}$.

10. ... $\frac{x^2 - y^2}{(x - y)^2}$ by $\frac{x^2 + xy}{x - y}$, ... $= \frac{1}{x}$.

RESOLUTION OF FRACTIONS INTO INFINITE SERIES.

142. Any proper fraction, with a compound denominator, can be resolved, by division, into an infinite series; for the numerator is a dividend, and the denominator is a divisor related to each other, as the remainder and divisor referred to in article (86); hence the quotient obtained, by dividing the first term of the numerator by that of the denominator, is not an integral quantity, but a fraction, and the process of division will never terminate, so that the quotient becomes an infinite series. The examples in Division that come under the observation in article (86), will all by this method produce an infinite series for a quotient. After a few of the terms of the quotient are found, the law of the series may be easily observed, and the succeeding terms can then be obtained without any further division.

When a law is thus found, by observing it to hold in every particular instance examined, it is said to be discovered by *induction*, which is the source of most discoveries in science. When a law is discovered by induction, its truth cannot always be implicitly relied on till it be directly demonstrated.

EXAMPLES.

1. Divide $a^2 + x^2$ by $a - x$.

$$a - x)a^2 + x^2(a + x + \frac{2x^2}{a} + \frac{2x^3}{a^2} +, \&c.$$

$$\begin{array}{l}
a^2 - ax \\
\hline
\quad ax + x^2 \\
\quad ax - x^2 \\
\hline
\qquad 2x^2 \\
\qquad 2x^2 - \dfrac{2x^3}{a} \\
\hline
\qquad\qquad \dfrac{2x^3}{a} \\
\qquad\qquad \dfrac{2x^3}{a} - \dfrac{2x^4}{a^2} \\
\hline
\qquad\qquad\qquad \dfrac{2x^4}{a^2}, \&c. \&c. \\
\hline
\end{array}$$

The law of the series is evident; for after the second term, if any term be multiplied by $\frac{x}{a}$, the product will be the succeeding term: this is by induction; but it is evident from the process of the division, that this must be the law of the series.

2. Reduce $\frac{a - x}{b + x}$ to an infinite series.

$$b + x)a - x\left(\frac{a}{b} - (a + b)\frac{x}{b^2} + (a + b)\frac{x^2}{b^3} -, \&c.\right.$$

$$\begin{array}{l}
a + \dfrac{ax}{b} \\
\hline
\quad -(a + b)\dfrac{x}{b} \\
\quad -(a + b)\dfrac{x}{b} - (a + b)\dfrac{x^2}{b^2} \\
\hline
\qquad\qquad (a + b)\dfrac{x^2}{b^2}, \&c. \\
\hline
\end{array}$$

3. Reduce $\dfrac{1}{3+1}$ to an infinite series.

$$\begin{array}{l} 3+1)1 \quad \left(\dfrac{1}{3}-\dfrac{1}{9}+\dfrac{1}{27}-, \text{\&c.}\right. \\ \quad 1+\dfrac{1}{3} \\ \hline \quad -\dfrac{1}{3} \\ \quad -\dfrac{1}{3}-\dfrac{1}{9} \\ \hline \quad \dfrac{1}{9}, \text{\&c.} \end{array}$$

Hence $$\frac{1}{3+1}=\frac{1}{3}-\frac{1}{3^2}+\frac{1}{3^3}-\frac{1}{3^4}+\ldots=\frac{1}{4}.$$

Similarly, $$\frac{1}{5-1}=\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\ldots=\frac{1}{4};$$

and generally $$\frac{1}{a-1}=\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\ldots$$

EXERCISES.

1. Divide a by $a+x$, $=1-\dfrac{x}{a}+\dfrac{x^2}{a^2}-\dfrac{x^3}{a^3}+\ldots$ to infinity.

2. ... 1 by $1-a$, . . . $=1+a+a^2+a^3+$, &c.

3. ... a by $x-b$, . $=\dfrac{a}{x}+\dfrac{ab}{x^2}+\dfrac{ab^2}{x^3}+\dfrac{ab^3}{x^4}+$, &c.

4. Resolve $\dfrac{x+2}{x+1}$ into an infinite series, $=1+\dfrac{1}{x}-\dfrac{1}{x^2}+\dfrac{1}{x^3}-$, &c.

5. ... $\dfrac{1}{3}$ or $\dfrac{1}{4-1}$, and $\dfrac{1}{2+1}$, into infinite series,

$=\dfrac{1}{4}+\dfrac{1}{4^2}+\dfrac{1}{4^3}+\ldots$ and $\dfrac{1}{2}-\dfrac{1}{2^2}+\dfrac{1}{2^3}-$, &c.

6. ... $\dfrac{ax}{a-x}$, . . . $=x+\dfrac{x^2}{a}+\dfrac{x^3}{a^2}+\dfrac{x^4}{a^3}+$, &c.

7. ... $\dfrac{1+x}{1-x}$, . . $=1+2x+2x^2+2x^3+$, &c.

8. ... $\dfrac{c+x}{c+y}$, $=1+(x-y)\dfrac{1}{c}-(x-y)\dfrac{y}{c^2}+(x-y)\dfrac{y^2}{c^3}-$, &c.

PRODUCTS OF NUMBERS—PRIME AND COMPOSITE NUMBERS.

I.—PRODUCTS OF NUMBERS.

The multiplier in the following theorems is the last number of the products.

143. THEOREM I. The sum of the products of several numbers by another number, is equal to the product of their sum by that number.

Or $ad + bd + cd = (a + b + c)d$, or $= sd$, if $s = a + b + c$.

For a, b, and c, taken d times, are just equal to s taken d times.

144. THEOREM II. The sum of the products of a number by several numbers, is equal to the product of that number by the sum of these numbers.

Or $da + db + dc = d(a + b + c)$, or $= ds$, if $s = a + b + c$.

For d taken as often as there are units in a, and b, and c, is just equal to d taken as often as there are units in s, or in the sum of a, b, and c.

145. THEOREM III. The product of two numbers is the same in whatever order they are taken.

Or $ab = ba$; that is, a taken b times, is equal to b taken a times; or a units, repeated as often as there are units in b, are equal to b units, repeated as often as there are units in a.

1st, When $b = 1$,

$$\text{Then } ab = ba, \text{ or } a \times 1 = 1 \times a;$$

for a units taken once is equal to 1 unit taken a times.

2d, When $a = b$, the theorem is evident.

3d, When a is a multiple of b, or $a = nb$,

Then $a.b = (b + b + \ldots n$ tin[illegible] $.b = bb$ taken n times by article (143), [illegible] $= nb.b$;

and $b.a = b(b + b + \ldots n \text{ times}) = bb$ taken n times by article (144), or $= b.nb$.

Hence $a.b = b.a$, and also $nb.b = b.nb$.

4th, When a contains b n times, with a remainder r, or $a = nb + r$,

Then $\qquad a \,.\, b = (nb + r)b = nb \,.\, b + r \,.\, b$ (by 143),

and $\qquad b \,.\, a = b(nb + r) = b \,.\, nb + b \,.\, r$ (by 144).

Now, $nb \,.\, b = b \,.\, nb$ (by 3d); and therefore, in order that $a \,.\, b = b \,.\, a$, this condition must exist, namely,

$$b \,.\, r = r \,.\, b$$

Now, $b \not> r$, let therefore $b = n'r + r'$, and treating b and r exactly in the same manner as a and b, and using n' for n, it may be shewn that the equality $r \,.\, b = b \,.\, r$ depends on the condition

$$r \,.\, r' = r' \,.\, r$$

Again, let $r = n''r' + r''$, and treating r and r' exactly in the same manner as a and b, and using n'' for n, it may be shewn that the equality $r \,.\, r' = r' \,.\, r$ depends on the condition that

$$r' \,.\, r'' = r'' \,.\, r'$$

It is evident that the quantities n, n', n'', and b, r, r', r'', are $=$ the quotients and divisors or remainders in performing on a and b, the process for finding the greatest common measure; hence some of the remainders must ultimately become either 1 or 0.

Suppose that $r'' = 1$, then $r' \,.\, r'' = r'' \,.\, r'$ (by 1st), and therefore $r \,.\, r' = r' \,.\, r$; and hence $a \,.\, b = b \,.\, a$.

Again, if $r'' = 0$, then the same conclusions evidently follow.

146. Theorem IV. The product of any number of factors is the same in whatever order they are taken.

Let there be n factors $a, b, c, d, \dots$; and first suppose that the product of $(n - 1)$ factors is the same in whatever order they are taken. The products of the n factors may be divided into various groups: *first*, those in which a is the first factor; *second*, those in which b is the first; *third*, those in which c is the first; and so on. Now, if P be the product of the $(n - 1)$ factors after the first or a, then any one of the first group is $= a \,.\, P$, since it is assumed that the product of $(n - 1)$ factors is the same in whatever order they are taken; and hence all the products in the first group are equal. So if P' be the product of $(n - 1)$ factors, excluding the second or b, any product in the second group is $= b \,.\, P'$, and they are therefore all equal. But some product in the second group must end in a, and will therefore be expressed by $P \,.\, a$; and since $a \,.\, P = P \,.\, a$ (145); therefore all the products in the first group are equal to those in the second. It may be similarly proved that all those in the second are equal to those in the third; and so on.

Therefore if the proposition be true for $(n-1)$ factors, it is also true for n factors. But it has been proved to be true for two factors (145), therefore it is true for three; hence therefore it is true for four; and generally therefore it is true for any number of factors.

147. Hence, the product of a number by that of other two numbers, is the same as the continued product of the three numbers taken in any order.

That is, $a \,.\, bc = abc$, or a multiplied by the product of bc is equal to the continued product of a, b, and c. For if $p = bc$, $a \,.\, bc = a \,.\, p = p \,.\, a = bc \,.\, a = bca$, and bca is the continued product of b, c, and a, which is the same in whatever order the factors a, b, c, are taken.

148. Hence also—a product is multiplied or divided by a number, if one of its factors be multiplied or divided by that number.

For it has been shewn that $a \,.\, bc = ab \,.\, c$; that is, if a be multiplied by the product bc, the result is equal to the product ab taken c times.

The case of dividing by a number is evident from this; for a and bc being the two factors, and c a third number, $a \,.\, \dfrac{bc}{c} = \dfrac{ab \,.\, c}{c} = ab.$

II.—PRIME AND COMPOSITE NUMBERS.

149. Theorem I. If a number is prime to other two numbers, it is prime to their product.

Let A, B, and P, be three numbers, P being prime to A and B, then P is prime to their product AB.

Let $B > P$, and perform on these two numbers the process for finding their greatest common measure (101); let $q, q', q'', \ldots$ be the successive quotients, and $R, R', R'', \ldots$ the corresponding remainders; then it is evident from that process that the following equations (103) are true.

$$\begin{array}{llll}
B = qP + R & \text{from which } AB & = qAP + AR & \ldots [1] \\
P = q'R + R' & \ldots \quad \ldots \quad AP & = q'AR + AR' & \ldots [2] \\
R = q''R' + R'' & \ldots \quad \ldots & = q''AR' + AR'' & \ldots [3] \\
R' = q'''R'' + R''' & \ldots \quad \ldots & = q'''AR'' + AR''' & \ldots [4] \\
\ldots \quad \ldots\ldots & \ldots & \ldots\ldots &
\end{array}$$

But since the successive remainders are continually diminishing, some one must at last be $= 0$ or 1. It cannot be 0, for then B and P would have a common measure, which is impossible, for they

are relatively prime; hence the remainder must $= 1$. Let the remainder that becomes $= 1$, be R''', and the last equations in the two ranks above become

$$R' = q'''R'' + 1, \quad \ldots \quad \ldots \quad AR' = q'''AR'' + A \ldots [5]$$

It appears, by equation [1], that the divisibility of AB by P depends on that of AR, for qAP is divisible by P; by equation [2], it appears that the divisibility of AR by P depends on that of AR' by P, for the first member being divisible by P, the second must be so; and if AR be so, $q'AR$ is so, and hence also AR': again, by equation [3], it appears that if AR and AR' are divisible by P, so is AR''; and by equation [4] or [5], since R''' is supposed $= 1$, it appears, if AR' and AR'' are divisible by P, that A must also be divisible by P. Thus, it appears that if B and P are prime, the divisibility of AB by P depends on the divisibility of A by P; and the divisibility of AB by any factor of P depends on that of A by this factor; and, consequently, when P is prime to A and B, it is prime to their product AB.

150. Cor. 1. If a number be prime to one factor of the product of two numbers, the divisibility of the product by the first number will depend on the divisibility of its other factor by that number.

For if P be prime to B, the divisibility of AB by P depends on that of A by P by [1].

151. Cor. 2. If a number be prime to one factor of a product, and if it exceed half the other factor, the product is not divisible by it.

For if P be prime to B, the divisibility of AB by P depends on that of A by P; but the greatest divisor of A is its half; therefore if P be greater than the half of A, it cannot divide AB.

152. Cor. 3. If a number be prime to several numbers, it is prime to their product.

Let p be prime to a, b, and c, it is prime to their product. For p is prime to ab, and if $ab = d$, p is prime to d and c, and therefore it is prime to their product dc or abc. And the proposition may be similarly proved in any other case.

153. Cor. 4. A composite number can be divided only by the factors of which it is composed; that is, either by its prime component factors, or the products of any number of them.

154. Cor. 5. If the factors of one composite number be prime to those of another, the two composite numbers are relatively prime.

155. Cor. 6. If two numbers are prime to each other, so are any powers of these numbers.

This corollary is evident from Cor. 5.

156. Cor. 7. If a fraction be in its lowest terms, the terms of any power of it are relatively prime.

Let $\frac{a}{b}$ be the fraction, then a is prime to b; hence the terms of the fraction $\frac{a^n}{b^n}$ are also relatively prime (Cor. 6).

157. Theorem II. If two numbers be prime, their product is their least common multiple.

Let p and q be prime, then if $m = pq$, m is their least common multiple. If it be not, let n be their least common multiple; then n is less than m, for pq is evidently a multiple of p and q. Since n is a multiple of p, it must be equal to the product of p by some number r; hence $n = pr$, and pr must be divisible by q, but

$$n \angle m, \text{ therefore } pr \angle pq, \text{ therefore } r \angle q;$$

and since q is prime to p, and greater than r, it is not a measure of pr or n (151). Hence n is not the least common multiple of p and q, therefore pq is so.

158. Theorem III. No root of an integer can be expressed by a vulgar fraction.

Let w be any integer, and if possible let its nth root be = the fraction $\frac{a}{b}$, which is expressed in its simplest form. Then $\sqrt[n]{w} = \frac{a}{b}$, and hence $w = \frac{a^n}{b^n}$; but a^n and b^n are prime to each other (156); and therefore the fraction $\frac{a^n}{b^n}$ cannot be = a whole number; hence the nth root of w cannot be $= \frac{a}{b}$.

PRODUCTS OF QUANTITIES—PRIME AND COMPOSITE QUANTITIES.

I.—PRODUCTS OF QUANTITIES.

159. Theorem I. The simple powers of all real algebraic quantities represent abstract numbers.

For if x be an algebraic quantity, it is admissible to involve it to any power, such as x^7. Now, x^7 means that x^6 is to be repeated x times (22); and therefore the factor x in the expression $x^7 = x^6 \times x$ must be an abstract number (6); but we never assign different values to the same symbol in the same expression; therefore the seven factors x in x^7 are abstract numbers. The same may be similarly proved of x^7, when x represents a compound quantity. It is hence evident that the proposition is generally true.

160. Since all real algebraic quantities, at least in their simple powers, denote abstract quantities, they ought, in order to represent concrete quantities, to be multiplied by the unit of measure.

161. Theorem II. The product of any two quantities is the same in whatever order the factors are taken.

Let A and B be two quantities, then A taken B times is equal to B taken A times.

1st, When A and B are simple quantities.

Since A and B represent numbers, the proposition is evidently true (145).

2d, When A and B are compound quantities.

In this case the quantities also representing numbers, it is evident from article (145), that the proposition is true respecting the numerical product; but the object here is to prove that the algebraic product is the same. Since in multiplying A by B, each term of A is successively multiplied by each term of B; and in multiplying B by A, each term of B is successively multiplied by each term of A; the terms of the two products must evidently be the same.

162. Theorem III. The product of any number of quantities is the same in whatever order they are taken.

Having proved the proposition for two quantities, it can be

proved for any number exactly in the same manner as for the products of numerical factors in article (146).

II.—PRIME AND COMPOSITE QUANTITIES.

163. All the propositions formerly proved respecting prime and composite numbers are evidently true in regard to prime and composite simple quantities.

For any two prime simple quantities must consist of different letters (97); and any two commensurable simple quantities must contain at least one common letter.

164. A quantity is said to be a *function* of one or of all the letters it contains, because its value depends on their values.

165. When the letters composing any function are connected only by the signs of the four fundamental operations, or of those of involution (176) or evolution (183), it is called an *algebraic* function.

166. When an algebraic function contains no fractions, it is said to be *integral;* and if it contains only integral powers of the letters, it is said to be *rational.*

167. When a function is arranged according to the powers of one of the letters, these powers being integral and positive, and their coefficients integral and rational, and either simple or compound, it is called an *entire integral* and *rational* function of that letter; and if the coefficients are any algebraic functions whatever, it is called an *integral* and *rational* function of that letter. Also, of two functions of the same quantity, that of the higher dimension is called the *higher.*

168. Theorem I. If an entire integral and rational function of a quantity be divisible by another, the product of the former by any other quantity will be divisible by the latter.

For the product of the first function by any term of any quantity will evidently be divisible by the second function; and hence the sum of the partial products—that is, the whole product—is divisible by it.

169. Theorem II. A measure of an entire integral and rational function of a quantity, which is independent of that quantity, is a measure of all its coefficients; and if the coefficient of the highest power of that quantity be unity, the coefficients have no measure.

Let the function $A + Bx + Cx^2 + \dots Mx^n$ have a divisor D independent of x, and let $a + bx + cx^2 + \dots mx^n$ be the quotient, then is

$$A + Bx + Cx^2 + \dots Mx^n = D(a + bx + cx^2 + \dots mx^n);$$

and hence, by the principle of undetermined coefficients (468),

$$A = Da,\ B = Db,\ C = Dc, \dots \mathrm{M} = Dm;$$

and therefore D is a factor of each of the coefficients $A, B, C, \dots M.$

170. Theorem III. If P and Q be two functions of x, and if the coefficients of either of them have a common measure, those of their product have the same common measure.

If the coefficients of neither P nor Q have a common measure, neither have those of their product.

Hence, when the coefficients of a function of x, which is the product of other two functions, have a common measure, so must the coefficients of one of the latter functions; and when the coefficients of the product have no common measure, neither have those of either of the factors.

If P and D' are two functions of x, such that the coefficients of P have no common measure, and those of D' have one, namely, m, so that $D' = mD$; then if D' be divisible by P, so is D.

If P and D be two functions of x, such that the coefficients of P have no common measure, and that P is prime to D, then P is also prime to mD, m being a simple integral quantity independent of x.

171. Theorem IV. Let P and B be two commensurable functions of x, P being the lower, and its coefficients not having a common measure, then if the process of (101) be performed on P and B, the last divisor is their greatest common measure; and this divisor is a function of x.

If the coefficients of P have a common measure r, so that $P=rP'$, then the greatest measure of P' and B found, as in (101), is also the greatest common measure of P and B, provided that r and the coefficients of B have no common measure. If they have, let it be n, so that $r = nr$, and $B = nB'$. Find C the greatest common measure of P' and B', and it will be the greatest common measure of $r'P'$ and B'; and hence nC will be the greatest common measure of $nr'P'$, and nB'— that is, of P and B.

172. Theorem V. If P, A, and B, be functions of x, and if P is prime to A and B, it is prime to their product.

173. The following theorems are also of importance :—

174. Theorem VI. The least common multiple of two integral functions that are relatively prime, is = their product.

Let the functions be $Ax^m + Bx^{m-1} + \ldots = M$, and $ax^n + bx^{n-1} + \ldots = N$, then MN is their least common multiple, and is of the degree $m + n$.

For if not, let $Sx^p + Tx^{p-1} + \ldots = P$ be their least common multiple, where $p = m + r$ and $r \angle n$. Then, since P contains M, it will be equal to the product of M by some polynomial $R = Wx^r + Xx^{r-1} + \ldots$; that is, $P = MR$, and is also divisible by N; and M and N being prime, R must be divisible by N (150), although N be of a higher dimension than R, which is impossible.

Again, if $p = m + n$, then $r = n$, and R being of the same dimension with N, and also divisible by it, it must be identical with it, or else the coefficients of R will be multiples of those of N, and exceed them in dimension. Therefore MN is the least common multiple of M and N.

175. When the division of one integral and rational function of a quantity (168) by another produces a quotient without a remainder, in which the coefficients of this quantity are not integral, the latter function is called a *relative* measure of the former; and a similar measure of two or more functions is called a *relative* common measure.

Thus, the greatest common measure of $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$ is $2x^3 - 4x^2 + x - 1$, whereas their greatest relative common measure is $\frac{39}{2}x^3 - 39x^2 + \frac{39}{4}x - \frac{39}{4}$.

If in any case the leading quantity in the first term of any dividend or remainder, in the process for finding the greatest relative common measure of two functions, is divisible by that in the first term of the divisor, while the coefficient of the former term is not divisible by that of the latter, the coefficient of the resulting term in the quotient will be fractional; for it is not necessary in finding this measure, in any case, to multiply the dividend by such a quantity as will make the coefficient of its first term divisible by that of the divisor.

INVOLUTION.

176. The object of *involution* is to involve, or raise quantities or roots, to any assigned powers (31). Powers are said to be *odd* or *even*, according as their exponents are odd or even numbers.

It appears from article 31 that any power of a quantity is equal to the continued product of that quantity, repeated as factor as often as there are units in the exponent of the power; but, generally, powers may be found by more concise methods.

CASE I.—TO INVOLVE SIMPLE QUANTITIES TO ANY ASSIGNED POWER.

177. Rule. Multiply the exponent of the quantity by that of the power.

If the quantity has a coefficient, involve it separately by actual multiplication (31), and prefix its power to that of the literal part.

Odd powers of negative quantities are negative; in other cases the powers are positive.

When the given quantity is the product of two or more factors, their powers must be found separately, and then the product of these separate powers is = the required power.

EXAMPLES.

1. The square of $a = (a)^2 = a^{1\times 2} = a^2$.

2. ... ... of $4a^3 = (4a^3)^2 = 4^2a^{3\times 2} = 16a^6$.

3. The cube of $-a^4 = (-a^4)^3 = -a^{4\times 3} = -a^{12}$.

4. ... ... of $-3xy^2 = (-3xy^2)^3 = -3^3x^3y^6 = -27x^3y^6$, for $(xy^2)^3 = (x)^3(y^2)^3 = x^3y^6$.

5. The fourth power of $5a^2x^3 = (5a^2x^3)^4 = 5^4a^8x^{12} = 625a^8x^{12}$.

6. The third power of $-2ax^2y^3 = (-2ax^2y^3)^3 = -2^3a^3x^6y^9 = -8a^3x^6y^9$.

7. The nth power of $a^m = (a^m)^n = a^{mn}$.

The principle on which the rule is founded is that $(a^3)^2 = a^3 . a^3 = a^{3\times 2} = a^6$; $(a^5)^3 = a^5 . a^5 . a^5 = a^{5\times 3} = a^{15}$; and generally $(a^n)^m = a^n . a^n . a^n \ldots a^n$ being repeated m times as a factor; but this product is a power of a, whose exponent is $= n + n + n \ldots n$ being repeated m times (65), which sum is $= mn$; or $(a^n)^m = a^{mn}$. Similarly $(x^3y^4)^2 = x^3y^4 . x^3y^4 = x^{3\times 2}y^{4\times 2} = x^6y^8$; so $(x^3y^4)^3 = x^9y^{12}$; and generally $(x^my^n)^r = x^my^n \times x^my^n \times \ldots x^my^n$ being repeated r times as a factor; but this product (65) is $= x^m . x^m . x^m \ldots \times y^n . y^n . y^n \ldots x^m$ and y^n being each repeated r times; and by the preceding proof, the product is $= x^{rm} \times y^{rm} = x^{rm}y^{rn}$. When the quantity consists of three simple factors, the proof is the same; and this comprehends also a numerical coefficient, since it may be considered as a simple factor.

The third part of the rule regarding the signs is evident from the rule for the signs in Multiplication (64). Thus, $(+x)(+x) = +x^2$ and $(-x)(-x) = +x^2$; also $(-x)(-x)(-x) = -x^3$, for $(-x)(-x) = x^2$, and $x^2(-x) = -x^3$. And when $-x$ is repeated as a factor an *odd* number of times, the product is *negative;* but if an *even* number of times, it is *positive.* And $+x$, repeated as a factor any number of times, whether odd or even, will always give a *positive* product. These results may be represented generally in the following manner:—Let n be an integer even or odd, then $2n$ is even, and $2n+1$ is odd, and

$$\left.\begin{aligned}
(+x)^{2n} &= \{(+x)^2\}^n = (+x^2)^n = +x^{2n} \\
(-x)^{2n} &= \{(-x)^2\}^n = (+x^2)^n = +x^{2n} \\
(+x)^{2n+1} &= +x(+x)^{2n} = +x(+x^{2n}) = +x^{2n+1} \\
(-x)^{2n+1} &= -x(-x)^{2n} = -x(+x^{2n}) = -x^{2n+1}
\end{aligned}\right\} A$$

The last only gives a negative result, since it is an *odd* power of a *negative* quantity. These expressions may be more briefly represented by taking $\pm a$ (plus or minus a) to denote $+a$ or $-a$, as may be required. When the upper sign is taken, the first line of (A) above will be given, and when the lower, the second line, if the exponent be $2n$; and when it is $2n+1$, the third line of (A) above will be given when the upper sign is taken, and the fourth when the lower sign is taken. The expressions are

$$(\pm x)^{2n} = \{(\pm x)^2\}^n = (+x^2)^n = +x^{2n}$$
$$(\pm x)^{2n+1} = \pm x(\pm x)^{2n} = \pm x(+x^{2n}) = \pm x^{2n+1}$$

EXERCISES.

1. Find the cube of $2x$, . . (the power) $P=8x^3$.
2. ... square of $3x^2y$, ... $P=9x^4y^2$.
3. ... cube of $-4a^3x$, ... $P=-64a^9x^3$.
4. ... cube of $-8x^3y^2$, ... $P=-512x^9y^6$.
5. ... fourth power of $7a^3x^5$, ... $P=2401a^{12}x^{20}$.
6. ... seventh power of $-2xy^4z^3$, ... $P=-128x^7y^{28}z^{21}$.

CASE II.—TO INVOLVE COMPOUND QUANTITIES TO ANY ASSIGNED POWER.

178. Rule. Involve the quantity to the proposed power by actual multiplication.

EXAMPLES.

1. The square of $a + x = (a+x)(a+x) = a^2 + 2ax + x^2$.

This result is found by multiplying $a+x$ by $a+x$ (31).

2. The square of $a - x = (a-x)(a-x) = a^2 - 2ax + x^2$.
3. The square of $ax + cy = (ax + cy)^2 = a^2x^2 + 2acxy + c^2y^2$.

179. It appears by inspecting the coefficients of the three terms in this result, which is found by actual multiplication, that $(2ac)^2 = 4a^2c^2$; and hence,

180. When a trinomial properly arranged (72) is a complete square, the square of the coefficient of the middle term is equal to four times the product of the coefficients of the extremes.

4. The cube of $x - y$.

By actually multiplying, it is found that $(x-y)^2 = (x-y)(x-y) = x^2 - 2xy + y^2$; and this being again multiplied by $x - y$, gives

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 = x^3 - y^3 - 3xy(x-y).$$

From which it appears that the cube of a binomial is = the cube of each of its terms, and three times their product into their sum, with their proper signs.

The rule is evident from the definition of powers (31), as it applies to all quantities, simple or compound.

EXERCISES.

1. Find the square of $1 - x$, . . . $= 1 - 2x + x^2$.
2. ... square of $x + 1$, . . . $= x^2 + 2x + 1$.
3. ... square of $ax - cy$, . $= a^2x^2 - 2acxy + c^2y^2$.
4. ... cube of $a - x$, . $= a^3 - 3a^2x + 3ax^2 - x^3$.
5. ... square of $2x^2 - 3y^2$, . $= 4x^4 - 12x^2y^2 + 9y^4$.
6. ... fourth power of $a - x$,
$$= a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4.$$
7. ... cube of $2x^2 - 3x + 1$,
$$= 8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1.$$
8. ... sixth power of $x - y$,
$$= x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6.$$
9. ... nth power of $2ax^m$, ... $= 2^na^nx^{mn}$.

181. Should the given quantity consist of two or more factors, and one or more of these factors be compound quantities, and should it be intended merely to express the power to which each factor is to be raised, the rule of the first case (177) must be applied to these factors, as if they were all simple quantities. Thus, we raise $2x^3(a - x)$ to the second power.

$$\{2x^3(a - x)\}^2 = 2^2x^6(a - x)^2.$$

But when the powers and products of the factors are not merely to be represented, but actually to be expressed by a series of simple quantities or monomials (that is, to be *expanded* or *developed*), the powers to which the separate factors are to be raised may be expressed, in the first place, as above, and afterwards, the operations indicated must be actually performed; thus, the quantity $2^2x^6(a - x)^2$ given above is $= 4x^6(a^2 - 2ax + x^2) = 4a^2x^6 - 8ax^7 + 4x^8$.

Or the given quantity may first be actually developed, and then the result raised to the given power. Thus, the given quantity $2x^3(a - x)$ when expanded is $= 2ax^3 - 2x^4$, which, being involved

to the second power by actual multiplication, gives $(2ax^3 - 2x^4)^2 = 4a^2x^6 - 8ax^7 + 4x^8$.

So $\{3x^3(a-x)^2(a+x)\}^2 = 3^2x^6(a-x)^4(a+x)^2 = 9x^6(a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4)(a^2 + 2ax + x^2) = 9x^6(a^6 - 2a^5x - a^4x^2 + 4a^3x^3 - a^2x^4 - 2ax^5 + x^6)$.

Now, actually multiplying each term by $9x^6$, the final polynomial is obtained. If the given quantity be first expanded, it gives $3x^3(a^2 - 2ax + x^2)(a + x) = 3x^3(a^3 - a^2x - ax^2 + x^3)$, and the square of this is $= 9x^6(a^3 - a^2x - ax^2 + x^3)^2$; and if the square of the second factor be actually found, it will be the same as the second factor above; and this being actually multiplied by $9x^6$, gives the final result.

Powers of binomials are very easily and concisely found by means of the binomial theorem given in article (482); and the square of a polynomial can easily be found as follows:—

Rule. To find the square of a polynomial — square the first term, and take twice its product into all the terms that come after it; then square the second term, and take twice its product into all the terms that come after it; in the same manner square all the other terms in succession, and take twice their product into all the terms that come after them up to the last; thus, $(a + b + c + d)^2 = a^2 + 2ab + 2ac + 2ad + b^2 + 2bc + 2bd + c^2 + 2cd + d^2$; which may be verified by actual multiplication.

CASE III.—TO INVOLVE FRACTIONAL QUANTITIES TO ANY POWER.

182. Rule. Involve the numerator and denominator to the required power, and the results will be respectively the same terms of the fraction, which is the required power.

EXAMPLES.

1. The square of $\dfrac{2x^3}{3y} = \left(\dfrac{2x^3}{3y}\right)^2 = \dfrac{(2x^3)^2}{(3y\ )^2} = \dfrac{4x^6}{9y^2}$.

2. ... cube of $-\dfrac{4a^2x^4}{5y^2z} = \left(-\dfrac{4a^2x^4}{5y^2z}\right)^3 = -\dfrac{(4a^2x^4)^3}{(5y^2z\ )^3} = -\dfrac{64a^6x^{12}}{125y^6z^3}$.

3. ... square of $\dfrac{2(a-x)}{3(a+x)} = \left(\dfrac{2(a-x)}{3(a+x)}\right)^2 = \dfrac{\{2(a-x)\}^2}{\{3(a+x)\}^2}$

$$= \frac{2^2(a-x)^2}{3^2(a+x)^2} = \frac{4(a^2 - 2ax + x^2)}{9(a^2 + 2ax + x^2)} = \frac{4a^2 - 8ax + 4x^2}{9a^2 + 18ax + 9x^2}.$$

4. The cube of $\frac{2a^2x(x-y)}{3y^4z^3} = \left(\frac{2a^2x(x-y)}{3y^4z^3}\right)^3 = \frac{2^3a^6x^3(x-y)^3}{3^3y^{12}z^9}$

$$= \frac{8a^6x^3(x^3 - 3x^2y + 3xy^2 - y^3)}{27y^{12}z^9}.$$

If the second factor of the numerator be then multiplied by $8a^6x^3$, it will produce a polynomial; but it is often more convenient to allow such an expression to remain in its present form.

The principle of the rule depends on the definition of a power (31), and the rule for multiplying fractions (137).

EXAMPLES.

1. Find the square of $\frac{3ax^2}{2y^3}$, ... $= \frac{9a^2x^4}{4y^6}$

2. ... the square of $-\frac{2x^2}{3y}$, ... $= \frac{4x^4}{9y^2}$

3. ... the cube of $-\frac{5a^2y^3}{4z}$, . . . $= -\frac{125a^6y^9}{64z^3}$

4. ... the square of $\frac{3(a-x)}{2(a+x)}$, . . $= \frac{9a^2 - 18ax + 9x^2}{4a^2 + 8ax + 4x^2}$

When the exponent of a power is a composite number, the power may be found by dividing the exponent into its factors, and then finding in succession the powers that are denoted by these factors.

Thus, $(a^2)^{12} = (a^2)^{3\times4} = (a^6)^4 = a^{24}$.

This method is the most concise when the power is to be found by actual multiplication, as, for example, when powers of numbers are to be found.

Thus, $(3)^4 = (3)^{2\times2} = (9)^2 = 81$.

$(2)^5 = (2)^4 \times 2 = (2^2)^2 \times 2 = 4^2 \times 2 = 16 \times 2 = 32$. Or thus, $2^2 = 4$, $4^2 = 16$, and $2^5 = 16 \times 2 = 32$.

EVOLUTION.

183. The operation of evolution is the reverse of that of involution, so that by this process it is intended to find a quantity such that a certain power of it shall be equal to the given quantity.

Thus, if a^6 be the given quantity, and it be required to find another whose square shall be $= a^6$, that quantity is evidently a^3, for $(a^3)^2 = a^6$.

The quantity to be found by evolution is called the *root* of the given quantity, and hence this process is also called the *extraction of roots.*

Thus, a^3, the quantity found above, is called the *square* or *second root* of a^6. So a^2 is the *cube* or *third root* of a^6, for $(a^2)^3 = a^6$. Likewise a^2 is the *fifth root* of a^{10}, for $(a^2)^5 = a^{10}$. Similarly a^m is called the nth root of a^{mn}, for $(a^m)^n = a^{mn}$ by the first case of involution. So a is the square root of a^2, or the cube root of a^3, or the fourth root of a^4, or generally the nth root of a^n.

184. The *exponent* of a root is the number that denotes it; and the root is said to be *odd* or *even*, according as its exponent is an odd or even number.

Thus, 2 with the radical sign ($\sqrt[2]{\ }$) is the exponent of the square root, 3 of the cube root ($\sqrt[3]{\ }$) (35), and n of the nth root ($\sqrt[n]{\ }$).

185. The denominators of fractional exponents are used to denote roots (35); thus, $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{n}}$, represent respectively the square root of a, the cube root of a, and the nth root of a.

CASE I.—TO EXTRACT THE ROOT OF A SIMPLE QUANTITY.

186. Rule. Divide the exponent of the quantity by that of the root.

If the quantity have a coefficient, its root is to be extracted by arithmetic.

When the given quantity is the product of two or more simple quantities, find the required roots of the latter quantities, and their product will be the required root.

The rule for the signs is, that *an even root of a positive quantity is either positive or negative; an odd root of a quantity has the same sign as the quantity itself;* and, *an even root of a negative quantity cannot be assigned.*

EXAMPLES.

1. The square root of $4a^4x^2 = (4a^4x^2)^{\frac{1}{2}} = 2a^{\frac{4}{2}}x^{\frac{2}{2}} = 2a^2x$.

2. ... cube root of $-27x^6y^9 = (-27x^6y^9)^{\frac{1}{3}} = -3x^{\frac{6}{3}}y^{\frac{9}{3}} = -3x^2y^3$.

3. ... nth root of $2^nx^ny^{2n}z^{mn} = (2^nx^ny^{2n}z^{mn})^{\frac{1}{n}} = \pm 2^{\frac{n}{n}}x^{\frac{n}{n}}y^{\frac{2n}{n}}z^{\frac{mn}{n}} = \pm 2xy^2z^m$.

In the first example, the sign may be either + or −; but in any particular case, circumstances must determine which of the signs is to be taken. In the last example, + or − is used on account of the uncertainty whether n be an odd or an even number. If it be odd, + only is to be used; but if even, either + or − may be prefixed.

The reason of the rule for extracting roots is evident from the reverse operation of involution. For $(a^m)^n = a^{mn}$, and hence a^m is the nth root of a^{mn}, that is, $a^m = \sqrt[n]{a^{mn}} = a^{\frac{mn}{n}}$; that is, the exponent of the given quantity (mn) is to be divided by that of the required root (n); also $\sqrt[r]{x^{mr}y^{nr}} = x^my^n$. The rule for the signs is also evident from the corresponding rule in involution (177), as it is simply the reverse. A positive power may have either a positive or negative even root, or a positive odd root; but a negative power can have only a negative odd root.

It appears by involution (177) that $\sqrt[2n]{x^{2n}} = \pm x$, and $(\pm x^{2n+1})^{\frac{1}{2n+1}} = \pm x$; in this last case, when the root is odd, the upper signs are taken together, and the lower signs together; or these results may be expressed thus, $\sqrt[2n]{x^{2n}} = +x$ or $-x$, $(x^{2n+1})^{\frac{1}{2n+1}} = +x$, and $(-x^{2n+1})^{\frac{1}{2n+1}} = -x$.

EXERCISES.

1. Extract the square root of $4x^2$, . . . $= \pm 2x$.
2. ... the cube root of $-8a^3x^6$, ... $= -2ax^2$.
3. ... the fourth root of $81a^4x^8y^4$, . . . $= 3ax^2y$.
4. ... the fifth root of $-32x^5y^{15}$, . . . $= -2xy^3$.
5. ... the nth root of $a^nx^{2n}y^{3n}$, . . . $= ax^2y^3$.
6. ... the rth root of $x^{rm}y^{2rn}$, ... $= x^my^{2n}$.

CASE II. — TO EXTRACT THE ROOT OF A FRACTION, WHOSE TERMS ARE SIMPLE QUANTITIES.

187. RULE. Extract the required roots of the terms of the given fraction, and these will be the corresponding terms of the fraction which is the required root.

EXAMPLES.

1. The square root of $\frac{4x^2y^6}{9z^4} = \sqrt{\frac{4x^2y^6}{9z^4}} = \frac{\sqrt{4x^2y^6}}{\sqrt{9z^4}} = \frac{2xy^3}{3z^2}$.

2. ... cube root of $-\frac{8x^6y^3}{27z^3} = \left(-\frac{8x^6y^3}{27z^3}\right)^{\frac{1}{3}} = -\frac{(8x^6y^3)^{\frac{1}{3}}}{(27z^3)^{\frac{1}{3}}} = -\frac{2x^2y}{3z}$.

3. ... nth root of $\frac{x^{rn}}{y^{sn}} = \sqrt[n]{\frac{x^{rn}}{y^{sn}}} = \frac{\sqrt[n]{x^{rn}}}{\sqrt[n]{y^{sn}}} = \frac{x^r}{y^s}$.

The rule may be proved thus:—By involution (182), $\left(\frac{x^m}{y^n}\right)^r = \frac{x^{mr}}{y^{nr}}$; and therefore the rth root of $\frac{x^{mr}}{y^{nr}}$ is $\frac{x^m}{y^n}$, that is, $\sqrt[r]{\frac{x^{mr}}{y^{nr}}} = \frac{x^m}{y^n} = \frac{\sqrt[r]{x^{mr}}}{\sqrt[r]{y^{nr}}}$.

EXERCISES.

1. Find the square root of $\frac{a^4x^2}{4y^2}$, ... = $\pm\frac{a^2x}{2y}$.

2. ... the cube root of $-\frac{8a^6y^9}{27x^3z^6}$, . . = $-\frac{2a^2y^3}{3xz^2}$.

3. ... the fourth root of $\frac{16a^4y^8}{81x^{16}z^4}$, ... = $\pm\frac{2ay^2}{3x^4z}$.

4. ... the fifth root of $\frac{32a^5x^{10}}{243y^5}$, . . . = $\frac{2ax^2}{3y}$.

5. ... the nth root of $\frac{x^ny^{2n}}{z^{3n}}$, ... = $\pm\frac{xy^2}{z^3}$.

6. ... the rth root of $\frac{a^rx^{nr}y^{2mr}}{c^{3r}z^{4r}}$, ... = $\pm\frac{ax^ny^{2m}}{c^3z^4}$.

The signs of the last two examples are doubtful, as n and r have no definite value. Were they both odd numbers, the sign would be $+$; and if even, the signs would be either $+$ or $-$.

188. Roots of quantities being considered as fractional powers, the rule for extracting roots of simple quantities is comprehended in the rule for involving them (177). Thus, if the square root be considered as the fractional power $\frac{1}{2}$, then

$$\sqrt{x^4} = (x^4)^{\frac{1}{2}} = x^{4\times\frac{1}{2}} = x^2;$$

in like manner $\sqrt[n]{x^{mn}} = (x^{mn})^{\frac{1}{n}} = x^{\frac{mn}{n}} = x^m;$

so that when roots are expressed as fractional powers, the rule becomes—*Multiply the exponent of the quantity by the exponent of the fractional power*—And the result is exactly the same as dividing by the exponent of the root. The preceding examples may be solved in this manner.

Roots, whose exponents are not prime numbers (97), may be extracted by successive operations. This is done by dividing the exponent into its factors, and extracting in succession the roots of which the factors are exponents. The following examples will illustrate this :—

Since $6 = 2 \times 3$, the 6th root will be extracted by first finding the square root of the given quantity, and then the cube root of this result; or by first finding the cube root of the quantity, and then the square root of this result. Thus—

$$\sqrt[6]{a^{12}} = \sqrt{(\sqrt[3]{a^{12}})} = \sqrt{(a^{12})^{\frac{1}{3}}} = \sqrt{a^4} = a^2,$$

or $$\sqrt[6]{a^{12}} = \sqrt[3]{(\sqrt{a^{12}})} = \sqrt[3]{(a^{12})^{\frac{1}{2}}} = \sqrt[3]{a^6} = a^2.$$

Likewise for the 8th root; since $8 = 2 \times 4$, or $= 4 \times 2$, or $= 2 \times 2 \times 2$, it may be found in three ways; thus—

$$\sqrt[8]{a^{16}} = \sqrt{(\sqrt[4]{a^{16}})} = \sqrt{(a^{16})^{\frac{1}{4}}} = \sqrt{a^4} = a^2,$$

or $$\sqrt[8]{a^{16}} = \sqrt[4]{(\sqrt{a^{16}})} = \sqrt[4]{(a^{16})^{\frac{1}{2}}} = \sqrt[4]{a^8} = a^2,$$

or $$\sqrt[8]{a^{16}} = \sqrt{\{\sqrt{(\sqrt{a^{16}})}\}} = \sqrt{(\sqrt{a^8})} = \sqrt{a^4} = a^2,$$

and $\sqrt[6]{64} = \sqrt[3]{(\sqrt{64})} = \sqrt[3]{8} = 2$, or $\sqrt[6]{64} = \sqrt{(\sqrt[3]{64})} = \sqrt{4} = 2.$

Generally, since $mnr = m \times n \times r$, the root, whose exponent is mnr, may be found by extracting in succession those whose exponents are m, n, and r; thus—

$$\sqrt[mnr]{x^{3mnr}} = \sqrt[m]{\{\sqrt[n]{(\sqrt[r]{x^{3mnr}})}\}} = \sqrt[m]{(\sqrt[n]{x^{3mn}})} = \sqrt[m]{x^{3m}} = x^3.$$

The reader may prove these results in the same way :—

$$\sqrt[10]{a^{20}} = a^2,\ \sqrt[6]{a^{18}} = a^3,\ \sqrt[12]{a^{24}} = a^2,\ \sqrt[mn]{a^{4mn}} = a^4.$$

Since $\sqrt{a^3}$ or $a^{\frac{3}{2}}$ means the square root of a^3, therefore the square of $\sqrt{a^3}$ or of $a^{\frac{3}{2}}$ is just a^3. So the square of $\sqrt{x^5}$ is x^5; the cube of $\sqrt[3]{x}$ is x; the nth power of $\sqrt[n]{x^m}$ is x^m; the square of $a\sqrt{x}$ is $(a)^2(\sqrt{x})^2 = a^2x$; the square of $\sqrt{-4}$ is -4; the square of $\sqrt{-a^2}$ is $-a^2$; and the square of $\sqrt{-1}$ is -1. Also the square of $\sqrt{(a-b)}$ is $a-b$.

CASE III.—TO EXTRACT THE SQUARE ROOT OF A COMPOUND QUANTITY.

189. RULE. Arrange the given quantity according to the powers of some of its letters (72), then the first term must be a complete square of some quantity. Find this quantity, which is the square root of the first term, and it will be the first term of the required root; write its square under the first term, and subtract it from the given quantity, bringing down only the second and third terms, and there will of course be no remainder under the first. Double the part of the root already found, and write it down as the first part of a divisor; divide by it the first term of the above remainder considered as a dividend, and the quotient will be the second term of the required root, which is also to be annexed as the second term of the divisor; multiply the divisor now formed by the last found term of the root, and subtract the product from the above remainder, and to this new remainder annex the next two terms of the given quantity for a new dividend. Proceed with the operation as before, and continue it till the given quantity is exhausted; or if there be always a remainder, the process will of course never terminate; the root will be an infinite series, and it can be carried out to as many terms as may be required.

EXAMPLES.

1. Find the square root of $4x^2 + 4a^2 - 8ax$.

Arranging the given quantity according to the powers of a, we have—

$$
\begin{array}{rl}
 & \sqrt{\{4a^2 - 8ax + 4x^2\}} = 2a - 2x \\
 & 4a^2 \\
\hline
4a - 2x \;\} & \quad -8ax + 4x^2 \\
 & \quad -8ax + 4x^2 \\
\hline
\end{array}
$$

2. Find the square root of $\frac{9}{16}x^4 - x^2y^2 + \frac{4}{9}y^4$.

$$\begin{array}{lll}
 & \sqrt{\{\frac{9}{16}x^4 - x^2y^2 + \frac{4}{9}y^4\}} = \frac{3}{4}x^2 - \frac{2}{3}y^2 \\
 & \frac{9}{16}x^4 \\ \hline
\frac{3}{2}x^2 - \frac{2}{3}y^2 \} & - x^2y^2 + \frac{4}{9}y^4 \\
 & - x^2y^2 + \frac{4}{9}y^4 \\ \hline
\end{array}$$

3. Find the square root of $4a^4 - 12a^3x + 13a^2x^2 - 6ax^3 + x^4$.

$$\begin{array}{ll}
 & \sqrt{\{4a^4 - 12a^3x + 13a^2x^2 - 6ax^3 + x^4\}} = 2a^2 - 3ax + x^2 \\
 & 4a^4 \\ \hline
4a^2 - 3ax \} & -12a^3x + 13a^2x^2 \\
 & -12a^3x + 9a^2x^2 \\ \hline
4a^2 - 6ax + x^2 & \} 4a^2x^2 - 6ax^3 + x^4 \\
 & 4a^2x^2 - 6ax^3 + x^4 \\ \hline
\end{array}$$

4. Find the square root of $a^2 - x^2$.

$$\begin{array}{ll}
 & \sqrt{\{a^2 - x^2\}} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} \\
 & a^2 \\ \hline
2a - \frac{x^2}{2a} \} & - x^2 \\
 & - x^2 + \frac{x^4}{4a^2} \\ \hline
2a - \frac{x^2}{a} - \frac{x^4}{8a^3} \} & - \frac{x^4}{4a^2} \\
 & - \frac{x^4}{4a^2} + \frac{x^6}{8a^4} + \frac{x^8}{64a^6} \\ \hline
2a - \frac{x^2}{a} - \frac{x^4}{4a^3} - \frac{x^6}{16a^5} \} & - \frac{x^6}{8a^4} - \frac{x^8}{64a^6} \\
 & - \frac{x^6}{8a^4} + \frac{x^8}{16a^6} + \frac{x^{10}}{64a^8} + \frac{x^{12}}{256a^{10}} \\ \hline
 & - \frac{5x^8}{64a^6} - \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}} \\ \hline
\end{array}$$

In this example the root is an infinite series, as the process would never terminate.

EXAMPLES.

Find the square roots of the following quantities:—

1. $(4x^4 - 12x^2y^2 + 9y^4)^{\frac{1}{2}}$, ... $= 2x^2 - 3y^2$.
2. $(9a^6 + 24a^3x^4 + 16x^8)^{\frac{1}{2}}$, ... $= 3a^3 + 4x^4$.
3. $(a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4)^{\frac{1}{2}}$, . $= a^2 - 2ax + x^2$
4. $\left(\frac{9}{16}a^4 - a^2z^2 + \frac{4}{9}z^4\right)^{\frac{1}{2}}$, ... $= \frac{3}{4}a^2 - \frac{2}{3}z^2$.
5. $(9a^4 - 12a^3b + 34a^2b^2 - 20ab^3 + 25b^4)^{\frac{1}{2}}$, $= 3a^2 - 2ab + 5b^2$.
6. $(49a^2x^2 - 24ax^3 + 25a^4 - 30a^3x + 16x^4)^{\frac{1}{2}}$, $= 5a^2 - 3ax + 4x^2$.
7. $(16a^4 - 40a^3x + 25a^2x^2 - 80ax^2y + 64x^2y^2 + 64a^2xy)^{\frac{1}{2}}$,

   $= 4a^2 - 5ax + 8xy$.
8. $(a^2 + x^2)^{\frac{1}{2}}$, . . . $= a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} -$, &c.

The rule for the extraction of the square root of a compound quantity may be easily found by observing by what means the terms of the square root of $a^2 + 2ax + x^2$ may be found. This quantity is the square of $a + x$, or $a + x$ is its square root. In this example, the first term a of the root is the square root of a^2, the first term of the given quantity. If the square of a (a^2) be now subtracted from the given quantity, the remainder is $2ax + x^2$, or $(2a + x)x$; and if the first term of this, or $2ax$, be divided by $2a$, which is the double of the part of the root already found, the quotient is x, the second term of the required root; and if this term x be added to $2a$, the divisor becomes $2a + x$; and this again being multiplied by x, gives $2ax + x^2$, which, taken from the above quantity, leaves no remainder.

When the given square is a trinomial, and its root therefore a binomial, the derivation of the rule is very simple; but it may be proved in a similar manner when the root is a trinomial or polynomial.

Let the root be $a + x + y$, then its square is $a^2 + 2ax + x^2 + 2ay + 2xy + y^2$; or if $a + x$ be denoted by a single letter c, then $a + x + y = c + y$, and $(a + x + y)^2 = (c + y)^2 = c^2 + 2cy + y^2$. Now,

by the former case, the square root $c + y$ can be found in the usual way, the operation being

$$\begin{array}{rl} & \sqrt{\{c^2 + 2cy + y^2\}} = c + y \\ (c)^2 = & c^2 \\ \hline 2c + y \;\} & 2cy + y^2 \\ & 2cy + y^2 \\ \hline \end{array}$$

But $c = a + x$, and $c^2 = a^2 + 2ax + x^2$; hence substituting their values for c and c^2, the above process is changed into

$$\begin{array}{rl} & \sqrt{\{(a^2 + 2ax + x^2) + 2(a + x)y + y^2\}} = (a + x) + y \\ (a + x)^2 = & a^2 + 2ax + x^2 \\ \hline 2(a + x) \quad + y \;\} & 2(a + x)y + y^2 \\ & 2(a + x)y + y^2 \\ \hline \end{array}$$

so that the same rule applies when the root is a trinomial.

To prove the rule when the root is a quadrinomial, as $a + x + y + z$, it is only necessary to assume $a + x + y = c$, and the root would then be $= c + z$, and the proof would be exactly similar to that of the preceding case; and it may be similarly proved that the rule is applicable to any polynomial.

190. It is evident also that

$$\sqrt{\{a^2x^2 \pm 2abxy + b^2y^2\}} = ax \pm by;$$

so that when a trinomial is a complete square, its square root is equal to those of its first and third terms connected by the sign $+$ or $-$, according as the second term of the trinomial is positive or negative.

The rule for extracting the square root of numbers may be easily proved by means of this rule, as it is only this rule slightly modified to adapt it to numbers.

Let it be required to find the square root of 1156. The root is 34, for $34^2 = 1156$, or $(30 + 4)^2 = 30^2 + 2 \times 30 \times 4 + 4^2$ (179), which is equal to $900 + 240 + 16$; the root of which is found by the preceding rule; thus—

$$\begin{array}{rl} & \sqrt{\{900 + 240 + 16\}} = 30 + 4 = 34 \\ 30^2 = & 900 \\ \hline 60 + 4 \;\} & 240 + 16 \\ & 240 + 16 \\ \hline \end{array} \qquad \begin{array}{rl} & \sqrt{11,56,} = 34 \\ 3^2 = & 9 \\ \hline 64 \;\} & 256 \\ & 256 \\ \hline \end{array}$$

The second form of the process in which the ciphers are omitted, is exactly according to the arithmetical rule for extracting the square root.

Take now the number 64516, which is the square of 254 or $(200 + 50 + 4)^2 = 40000 + 24500 + 16$, and

$$\begin{array}{lll}
 & \sqrt{\{40000 + 24500 + 16\}} = 200 + 50 + 4 & \\
200^2 = 40000 & & \\
\hline
400 \ + 50 & \}\ 24500 & \\
 & }\ 22500 & \\
\hline
500 \ + \ 4 & \}\ 2000 + 16 & \\
 & }\ 2000 + 16 & \\
\hline
\end{array}$$

Or thus :—

$$\begin{array}{ll}
 & \sqrt{6,45,16} = 254 \\
 & 2^2 = 4 \\
\hline
45 & \}\ 245 \\
 & }\ 225 \\
\hline
504 & \}\ 2016 \\
 & }\ 2016 \\
\hline
\end{array}$$

The process of extracting the square root of the number 64516 or $40000 + 24500 + 16$ is given above in two ways; the second is the usual method. The reason of the rule for dividing the given number into periods of two figures is evident, for there are four ciphers neglected after the 4 under 6 in the second process, and two are neglected after the 225; and these ciphers depend on the fact, that the nominal value of figures increases 10 times by a change of one place from right to left, and therefore their squares increase 100 times. Thus, in 45, the nominal value of 4 is $4 \times 10 = 40$, and its square $= 40^2 = 1600$; whereas 4 in the number 24 is $= 4$ units, and its square $= 4^2 = 16$.

CASE IV.—TO EXTRACT THE CUBE ROOT OF A COMPOUND QUANTITY.

191. RULE. Arrange the given quantity in descending order (72); then find the cube root of the first term, which will be the first term of the required root; place the cube of this first term of the root under the first term of the power. Take down the next three terms for a dividend, and for the first term of the divisor, multiply by three the square of the part of the root found; divide the first term of the dividend by this term, and the quotient is the second term of the root. To complete the divisor, annex the quantity last found to three times the root formerly found, and multiply this sum by the quantity last found; the product annexed to the former divisor gives the complete divisor. Multiply the completed divisor by the last found term of the root,

and subtract the product from the former dividend. To this remainder add some of the next terms of the given quantity for a new dividend, and continue the same process till the given quantity be exhausted, or till as many terms of the root be found as are required.

EXAMPLES.

1. Find the cube root of $a^3 + 3a^2x + 3ax^2 + x^3$

$$\begin{array}{rl} & \sqrt[3]{\{a^3 + 3a^2x + 3ax^2 + x^3\}} = a + x \\ (a)^3 = & a^3 \\ \hline 3a^2 + 3ax + x^2 \,\} & 3a^2x + 3ax^2 + x^3 \quad = \{3a^2 + (3a + x)x\}x \\ & 3a^2x + 3ax^2 + x^3 \\ \hline \end{array}$$

2. Find the cube root of $a^3 + 3a^2x + 3ax^2 + x^3 + 3a^2z + 6axz + 3x^2z + 3az^2 + 3xz^2 + z^3$.

$$\begin{array}{rl} & \sqrt[3]{\{a^3 + 3a^2x + 3ax^2 + x^3 + 3a^2z + 6axz + 3x^2z + 3az^2 + 3xz^2 + z^3\}} = a + x + z \\ (a)^3 = & a^3 \\ \hline 3a^2 + 3ax + x^2 \,\} & + 3a^2x + 3ax^2 + x^3 \\ & + 3a^2x + 3ax^2 + x^3 \\ \hline 3a^2 + 6ax + 3x^2 + 3az + 3xz + z^2 \,\} & 3a^2z + 6axz + 3x^2z + 3az^2 + 3xz^2 + z^3 \\ \text{or } 3(a + x)^2 + 3(a + x)z + z^2 & 3a^2z + 6axz + 3x^2z + 3az^2 + 3xz^2 + z^3 \\ \hline \end{array}$$

The second divisor here is formed according to the rule, by considering $(a + x)$ as a single term, as is shewn in the second form of this divisor; and then the divisor in this form being expanded, gives the first form of it.

The principle of the rule is evident, by merely considering that the given quantity in the first example is the cube of $a + x$, or $a + x$ is its cube root; and the rules for making up this trial and complete divisor are both obvious from the second form of the remainder. Also in the second example, after finding the first two terms of the root or $a + x$, the first term of the next divisor, formed according to the rule, is $3(a + x)^2$, or $3a^2 + 6ax + 3x^2$; and $3a^2z$ being divided by $3a^2$, gives z for the third term of the root; and hence the next term of the same divisor is $\{3(a+x)+z\}z$. The rule is thus applied to a trinomial, and it may evidently be applied generally, or when the root is a polynomial.

EXAMPLES.

Find the cube root of the following quantities:—

1. $(a^3 - 3a^2x + 3ax^2 - x^3)^{\frac{1}{3}}$, ... $= a - x$.
2. $(8a^6 - 36a^4x^2 + 54a^2x^4 - 27x^6)^{\frac{1}{3}}$, . . $= 2a^2 - 3x^2$.
3. $(x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1)^{\frac{1}{3}}$, $= x^2 - 2x + 1$.
4. $(27a^3 - 54a^2z + 36az^2 - 8z^3)^{\frac{1}{3}}$, . . $= 3a - 2z$.
5. $(1 - y^3)^{\frac{1}{3}}$, . . . $= 1 - \frac{y^3}{3} - \frac{y^6}{9} - \frac{5y^9}{81}$—, &c.

The arithmetical rule for extracting the cube root is derived from the algebraic rule. As an example, we may extract the cube root of 14706125. Its cube root is 245 or $200 + 40 + 5$. To adapt this to the second example given under the rule, let $200 = a$, $40 = x$, and $5 = z$.

$$\begin{array}{lll}
 & & \sqrt[3]{14706125} = 200 + 40 + 5 \\
 & a^3 = 200^3 = & 8000000 \\
3a^2 = 3 \times 200^2 & = 120000 & \}6706125 \\
(3a+x)x = (3 \times 200 + 40) \times 40 & = 25600 & \\
\text{then } 3a^2 + 3ax + x^2 & = 145600 & \}5824000 \\
3(a+x)^2 = 3 \times 240^2 & = 172800 & }882125 \\
\{3(a+x)+z\}z = (3 \times 240 + 5)5 & = 3625 & \\
\text{then } 3(a+x)^2 + 3(a+x)z + z^2 & = 176425\} & }882125
\end{array}$$

The second *trial* divisor 172800 is $= 3 \times 240^2$; but this is found more easily by placing the square of the quantity last found under the complete divisor; then adding this square, the complete divisor, and its second part into one sum, which will

give the next trial divisor, that is, $1600 + 145600 + 25600$. The reason of this is, that $3(a + x)^2 = 3a^2 + 6ax + 3x^2 = (3a^2 + 3ax + x^2) + (3a + x)x + x^2$, which is precisely the former complete divisor with its second part, together with the square of the part last found.

This method of finding the trial divisor is very simple compared with the common method, especially when the root extends to more than three places; hence it is given in some treatises on arithmetic.

The preceding process, freed from the unnecessary ciphers, may be thus stated—

$$
\begin{array}{lrl}
 & & \sqrt[3]{14,706,125} = 245 \\
 & 2^3 = & 8 \\
300 \times 2^2 & = 1200 & \}6706 \\
(30 \times 2 + 4)4 & = 256 & \\
 & 1456 \times 4 & = 5824 \\
300 \times 24^2 & = 172800 & \}882125 \\
(30 \times 24 + 5)5 & = 3625 & \\
 & 176425 \times 5 & = 882125
\end{array}
$$

The reason for dividing the given number into periods of three figures each, is evident from this example. It is evident, too, that 2^3 or 8 is the next lower cube number to 14, for $3^3 = 27$. The nominal value of the 2, however, is 200, and its cube contains 6 ciphers, which would extend over two periods. The periods must consist of three places, for the nominal value of a figure increasing 10 times when it is changed one place to the left, its cube increases 10^3 or 1000 times.

CASE V.—TO EXTRACT ANY ROOT OF A COMPOUND QUANTITY.

192. RULE. Arrange the quantity as in the two preceding cases; then find the required root of the first term, and this will be the first term of the root. Involve this term to the power corresponding to the root, and place this power under the first term of the given quantity to which it will be equal; then subtract it from the given quantity, and the remainder will be a dividend. Involve the first term of the root to a power whose exponent is one less than that of the root, and multiply this power by the exponent of the root, the product will be a *constant* divisor. Divide the first term of the remainder by this divisor, and the quotient will be the next term of the root. Involve the part of the root now found to the corresponding power, and subtract this power from the original quantity, and the

remainder will be a new dividend. Divide its first term by the constant divisor as before, and continue the same process as far as may be necessary.

EXAMPLES.

1. Find the fourth root of $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$.

$$\sqrt[4]{\{a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4\}} = a - x$$
$$(a)^4 = a^4$$
$$\overline{4a^3\} - 4a^3x}$$
$$(a - x)^4 = a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$$

Methods for extracting the fourth and higher roots of compound quantities, and also of numbers, similar to those for the square and cube root, might be deduced; but they would be very complex in their application.

The foundation of the rule will appear by taking a particular example. Let $a + b + c + d$ be raised to the fourth power. This power may be expressed thus (494),

$$(a + b + c + d)^4 = \{(a + b + c) + d\}^4$$
$$= (a + b + c)^4 + 4(a + b + c)^3d +, \&c.;$$

and $$(a + b + c)^4 = \{(a + b) + c\}^4$$
$$= (a + b)^4 + 4(a + b)^3c +, \&c.;$$

also $$(a + b)^4 = a^4 + 4a^3b +, \&c.:$$

so that the fourth power of $a + b + c + d$, may be thus expressed,

$$a^4 + 4a^3b + \ldots + 4a^3c + \ldots + 4a^3d + \ldots, \&c.$$

Or it may be represented in these three forms:—

$$a^4 + 4a^3b + \ldots, \ (a + b)^4 + 4a^3c + \ldots, \ (a + b + c)^4 + 4a^3d + \ldots$$

neglecting some of the terms which it is unnecessary to put down. After subtracting a^4, the first term of the remainder is $4a^3b$; and this being divided by the constant divisor $4a^3$, gives b, the second term of the root. After subtracting $(a + b)^4 = a^4 + 4a^3b + \ldots$, the first term of the remainder is $4a^3c$; and this being divided by $4a^3$, gives c, the third term of the root. Again, after subtracting $(a + b + c)^4 = a^4 + 4a^3b + \ldots$, the first term of the remainder is $4a^3d$; and this being divided by $4a^3$, gives d, the fourth term of the root. By the same method the rule can be proved in any other case.

EXERCISES.

1. Find the fourth root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$, ... $= 2a - 3x$.

2. ... the cube root of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$, ... $= x^2 - 2x + 1$.

3. ... the fifth root of $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$, ... $= 2x - 1$.

4. ... the fourth root of $a^{4m} - 4a^{3m}x^n + 6a^{2m}x^{2n} - 4a^mx^{3n} + x^{4n}$, ... $= a^m - x^n$.

IRRATIONAL QUANTITIES.

193. When any proposed root of a quantity cannot be exactly assigned, this root of it is called an *irrational quantity*, a *radical*, or a *surd*.

Thus, the square root of 2 or $\sqrt{2} = 1{\cdot}4142\ldots$, or $= 1$, with an interminate decimal fraction $\cdot4142\ldots$, which is not a periodic decimal; or, in other words, $\sqrt{2}$ is an interminate number, the value of which cannot be expressed by any integer or vulgar fraction (158); hence it is a *surd*. So $\sqrt[3]{2}$, $\sqrt[4]{3}$, $\sqrt{8}$, $\sqrt{a}$, $\sqrt[3]{a^2}$, $\sqrt{(a+x)}$, are *irrational* quantities.

194. An *interminate* number is a non-periodic interminate decimal, either with or without a prefixed integer; and hence it cannot be expressed by either an integer or a vulgar fraction; a *terminate* number is either an integer, a vulgar fraction, or a mixed number.

195. All irrational numbers are interminate; but interminate numbers are not all irrational; for irrational numbers are only a particular kind of interminate numbers.

196. An irrational root of any integral quantity cannot be expressed by a fractional quantity (158).

197. Irrational quantities may be expressed either by using the radical sign, or a fractional exponent whose numerator denotes the power to which the letter is raised, and whose denominator indicates the root to be extracted.

Thus, $\sqrt{a} = a^{\frac{1}{2}}$, $\sqrt[3]{a^2} = a^{\frac{2}{3}}$, $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, and $\sqrt[n]{(a^2 + x^2)^m} = (a^2 + x^2)^{\frac{m}{n}}$.

198. *Rational* quantities are such as have integral exponents.

Thus, a, $3ax^4$, $2(a^2 - x^2)$, and $\dfrac{3a(a-x)}{4(a+x)^3}$, are rational quantities.

199. A quantity is said to be *rationalised* when it is reduced to the form of a rational quantity.

200. Surds are said to be of the *same denomination* when they are under the same radical sign.

201. Surds are said to be *similar* when they consist of the same quantity under the same radical sign.

CASE I.—TO REDUCE A RATIONAL QUANTITY TO THE FORM OF A SURD.

202. RULE. Raise the quantity to that power denoted by the exponent of the proposed surd, and then place this result under the corresponding radical sign or fractional exponent.

EXAMPLES.

1. Reduce 3 to the form of the square root.

$$(3)^2 = 3^2 = 9\text{; hence } 3 = \sqrt{9}\text{, or } = 9^{\frac{1}{2}}.$$

2. ... 2 to the form of the cube root.

$$(2)^3 = 2^3 = 8\text{; hence } 2 = \sqrt[3]{8}\text{, or } = 8^{\frac{1}{3}}.$$

3. ... a to the form of the fourth root.

$$(a)^4 = a^4\text{; hence } a = \sqrt[4]{a^4}\text{, or } = a^{\frac{4}{4}}.$$

4. ... a^3 to the form of the square root.

$$(a^3)^2 = a^{3\times 2} = a^6\text{; hence } a^3 = \sqrt{a^6}\text{, or } = a^{\frac{6}{2}}.$$

5. ... $2a$ to the form of the square root.

$$(2a)^2 = 4a^2\text{; hence } 2a = \sqrt{4a^2}\text{, or } = (4a^2)^{\frac{1}{2}}.$$

6. ... $\dfrac{2a^3}{3x}$ to the form of the square root.

$$\left(\frac{2a^3}{3x}\right)^2 = \frac{4a^6}{9x^2}\text{; hence } \frac{2a^3}{3x} = \sqrt{\frac{4a^6}{9x^2}}.$$

7. ... a^m to the form of the nth root.

$$(a^m)^n = a^{mn}\text{; hence } a^m = \sqrt[n]{a^{mn}}\text{, or } = (a^{mn})^{\frac{1}{n}}.$$

8. Reduce $\frac{a^m x^n}{y}$ to the form of the rth root.

$$\left(\frac{a^m x^n}{y}\right)^r = \frac{a^{mr} x^{nr}}{y^r}; \text{ hence } \frac{a^m x^n}{y} = \sqrt[r]{\frac{a^{mr} x^{nr}}{y^r}}, \text{ or } = \left(\frac{a^{mr} x^{nr}}{y^r}\right)^{\frac{1}{r}}.$$

9. Reduce $(a^2 - x^2)^m$ to the form of the nth root.

$$\{(a^2 - x^2)^m\}^n = (a^2 - x^2)^{mn}; \text{ hence } (a^2 - x^2)^m = \sqrt[n]{(a^2 - x^2)^{mn}},$$

$$\text{or } = \{(a^2 - x^2)^{mn}\}^{\frac{1}{n}}.$$

EXERCISES.

1. Reduce 5 to the form of the square root, . $= (25)^{\frac{1}{2}}$.
2. ... $2a^3$ ... ... $= (4a^6)^{\frac{1}{2}}$.
3. ... a^2x^4 to the form of the cube root, . $= (a^6x^{12})^{\frac{1}{3}}$.
4. ... $\frac{ax^2}{cy^3}$ to the form of the fourth root, . $= \left(\frac{a^4x^8}{c^4x^{12}}\right)^{\frac{1}{4}}$.
5. ... $a^m x^n$ to the form of the rth root, . $= (a^{mr}x^{nr})^{\frac{1}{r}}$.
6. ... $(a^2 - x^2)^2$ to the form of the cube root, $= \{(a^2 - x^2)^6\}^{\frac{1}{3}}$.
7. ... $\frac{a^2y^4}{2x^3}$ to the form of the nth root, . $= \left(\frac{a^{2n}y^{4n}}{2^n x^{3n}}\right)^{\frac{1}{n}}$.
8. ... $a^2x^3(x - y)^2$ to the form of the cube root,

$$= \{a^6x^9(x - y)^6\}^{\frac{1}{3}}.$$

The reason of the rule is evident; for a quantity being raised to any power, and then placed under the corresponding radical sign, its value is not altered.

CASE II.—TO REDUCE SURDS TO THEIR SIMPLEST FORM.

203. Rule. Divide the irrational quantity into two factors, one of which is a power of the same name as the root to be extracted, and prefix its root to the remaining radical factor.

EXAMPLES.

1. Reduce $\sqrt{8}$ to its most simple form.

$$\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}.$$

2. ... $\sqrt{6a^3}$ to its simplest form.

$$\sqrt{6a^3} = \sqrt{a^2}\sqrt{6a} = a\sqrt{6a}.$$

3. Reduce $\sqrt[3]{16a^5x^4}$ to its simplest form.

$$\sqrt[3]{16a^5x^4} = \sqrt[3]{8a^3x^3}\sqrt[3]{2a^2x} = 2ax\sqrt[3]{2a^2x}.$$

4. Simplify the fraction $\sqrt{\frac{192}{125}}$.

$$\sqrt{\frac{192}{125}} = \sqrt{\frac{64}{25}}\sqrt{\frac{3}{5}} = \frac{8}{5}\sqrt{\frac{3}{5}}.$$

5. Reduce $\sqrt{18a^2x^3(a^2-x^2)^5}$ to its simplest form.

$$\sqrt{18a^2x^3(a^2-x^2)^5} = \sqrt{9a^2x^2(a^2-x^2)^4}\sqrt{2x(a^2-x^2)} = 3ax(a^2-x^2)^2 \times \sqrt{2x(a^2-x^2)}.$$

6. Reduce $\sqrt{\frac{8a^3x^2y^2}{32c^4y^3}}$ to its simplest form.

First simplifying the fraction, it becomes $\frac{a^3x^2}{4c^4y}$, and $\sqrt{\frac{a^3x^2}{4c^4y}} = \frac{\sqrt{a^2x^2}\sqrt{a}}{\sqrt{4c^4}\sqrt{y}} = \frac{ax\sqrt{a}}{2c^2\sqrt{y}} = \frac{ax}{2c^2}\sqrt{\frac{a}{y}}.$

7. Reduce $\sqrt[3]{\frac{16a^5x^4}{81c^4y^7}}$ to its simplest form.

$$\sqrt[3]{\frac{16a^5x^4}{81c^4y^7}} = \frac{\sqrt[3]{8a^3x^3}\sqrt[3]{2a^2x}}{\sqrt[3]{27c^3y^6}\sqrt[3]{3cy}} = \frac{2ax}{3cy^2}\sqrt[3]{\frac{2a^2x}{3cy}}.$$

The reason of the rule is, that any root of a quantity is equal to the product of the same root of its factors (186.)

EXERCISES.

1. Reduce $\sqrt{18}$ to its simplest form, . . $= 3\sqrt{2}$.
2. ... $\sqrt[3]{\frac{16}{81}}$ ... . $= \frac{2}{3}\sqrt[3]{\frac{2}{3}}$.
3. ... $\sqrt{8a^5x^2}$ ... . $= 2a^2x\sqrt{2a}$.
4. ... $\sqrt{\frac{8a^4x^3}{27c^6y^5}}$ ... . $= \frac{2a^2x}{3c^3y^2}\sqrt{\frac{2x}{3y}}$.
5. ... $\sqrt{a^5(a^6-a^4x^2)}$ ... . $= a^4\sqrt{a(a^2-x^2)}$.
6. ... $\sqrt[3]{16y^5(a^3x^4-x^7)}$ ... $= 2xy\sqrt[3]{2xy^2(a^3-x^3)}$.

CASE III.—TO TRANSFORM A FRACTION WITH A RADICAL DENOMINATOR TO AN EQUIVALENT FRACTION WITH A RATIONAL DENOMINATOR.

204. RULE I. When the quantity under the radical sign in the denominator consists of a single letter, find the least multiple of the exponent of the root that exceeds the exponent of the quantity under the radical sign; then multiply this quantity by such a power of the simple letter as will make its exponent equal to the preceding multiple; multiply the numerator by the same quantity placed under the radical sign, and extract the root of the denominator.

EXAMPLES.

1. Rationalise the denominator of $\frac{1}{\sqrt[3]{a^7}}$.

The least multiple of 3, that exceeds 7, is 9; hence multiply a^7 by a^2, and the numerator by $\sqrt[3]{a^2}$, then

$$\frac{1}{\sqrt[3]{a^7}} = \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^7a^2}} = \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^9}} = \frac{\sqrt[3]{a^2}}{a^3}.$$

2. Rationalise the denominator of $\frac{4}{2\sqrt[3]{12}}$.

The exponent of 12 is 1, and the least multiple of 3 above 1 is 3 itself; hence multiply by 12^2 or 144 under the sign $\sqrt[3]{\ }$, and

$$\frac{4}{2\sqrt[3]{12}} = \frac{4\sqrt[3]{144}}{2\sqrt[3]{1728}} = \frac{4\sqrt[3]{144}}{2\times12} = \frac{\sqrt[3]{144}}{6} = \frac{\sqrt[3]{8}\sqrt[3]{18}}{6} = \frac{2\sqrt[3]{18}}{6} = \frac{1}{3}\sqrt[3]{18}.$$

3. Rationalise the denominator of $\frac{4a}{3\sqrt[4]{x^5}}$.

The least multiple of 4 that exceeds 5 is 8; hence multiplying by $\sqrt[4]{x^3}\frac{4a\sqrt[4]{x^3}}{3\sqrt[4]{x^5x^3}} = \frac{4a\sqrt[4]{x^3}}{3\sqrt[4]{x^8}} = \frac{4a\sqrt[4]{x^3}}{3x^2}$.

4. Rationalise the denominator of $\frac{2\sqrt{a}}{5\sqrt[3]{x^{10}}}$.

The least multiple of 3 above 10 is 12; hence

$$\frac{2\sqrt{a}}{5\sqrt[3]{x^{10}}} = \frac{2\sqrt{a}\sqrt[3]{x^2}}{5\sqrt[3]{x^{10}x^2}} = \frac{2\sqrt{a}\sqrt[3]{x^2}}{5\sqrt[3]{x^{12}}} = \frac{2\sqrt{a}\sqrt[3]{x^2}}{5x^4}.$$

5. Rationalise the denominator of $\frac{2a}{\sqrt[n]{x^m}}$.

As the values of m and n are not known, the least known multiple of n that exceeds m is mn; hence multiply both terms of the fraction by $\sqrt[n]{x^{m(n-1)}}$, then

$$\frac{2a}{\sqrt[n]{x^m}} = \frac{2a\sqrt[n]{x^{m(n-1)}}}{\sqrt[n]{x^m x^{m(n-1)}}} = \frac{2a\sqrt[n]{x^{m(n-1)}}}{\sqrt[n]{x^{mn}}} = \frac{2a\sqrt[n]{x^{m(n-1)}}}{x^m}.$$

The reason of the rule is, that the value of a fraction is unchanged by its numerator and denominator being multiplied by the same quantity. The object of the rule is to make the quantity under the radical in the denominator a complete power corresponding to the root; the multiplier may sometimes be easily found by inspection.

EXERCISES.

1. Rationalise the denominator of $\frac{1}{\sqrt[3]{x^5}}$, . $= \frac{\sqrt[3]{x}}{x^2}$.

2. ... ... $\frac{3}{\sqrt{5}}$, . . $= \frac{3\sqrt{5}}{5}$.

3. ... ... $\frac{3x}{2\sqrt[4]{a^9}}$, . $= \frac{3x\sqrt[4]{a^3}}{2a^3}$.

4. ... ... $\frac{4\sqrt{a^3}}{\sqrt[3]{y^5}}$, . . $= \frac{4\sqrt{a^3}\sqrt[3]{y}}{y^2}$.

5. ... ... $\frac{1}{\sqrt{x^n}}$, . $= \frac{\sqrt{x^n}}{x^n}$.

6. ... ... $\frac{3}{2\sqrt[n]{x^{2m}}}$, . $= \frac{3\sqrt[n]{x^{m(n-2)}}}{2x^m}$.

205. RULE II. When the quantity under the radical sign in the denominator is the product of two or more factors; find a multiplier for each factor as in the former case, and multiply both terms of the fraction by their product, and then find the required root of the denominator.

EXAMPLES.

1. Rationalise the denominator of $\frac{1}{\sqrt[3]{a^4x^2}}$.

The multiplier is evidently a^2x under the sign $\sqrt[3]{\ }$; hence

$$\frac{1}{\sqrt[3]{a^4x^2}} = \frac{\sqrt[3]{a^2x}}{\sqrt[3]{a^4x^2a^2x}} = \frac{\sqrt[3]{a^2x}}{\sqrt[3]{a^6x^3}} = \frac{\sqrt[3]{a^2x}}{a^2x}.$$

2. Rationalise the denominator of $\dfrac{2a}{3\sqrt[r]{x^m y^n}}$.

The multiplier is $x^{m(r-1)}y^{n(r-1)}$ under the sign $\sqrt[r]{}$; and hence

$$\frac{2a}{3\sqrt[r]{x^m y^n}} = \frac{2a\sqrt[r]{x^{m(r-1)}y^{n(r-1)}}}{3\sqrt[r]{x^{mr}y^{nr}}} = \frac{2a\sqrt[r]{(x^m y^n)^{r-1}}}{3x^m y^n}.$$

EXERCISES.

1. Rationalise the denominator of $\dfrac{2a}{\sqrt[3]{x^4 y^5}}$, . $= \dfrac{2a\sqrt[3]{x^2 y}}{x^2 y^2}$.

2. ... ... $\dfrac{3x^2}{\sqrt{a^3 y z^5}}$, . $= \dfrac{3x^2\sqrt{ayz}}{a^2 y z^3}$.

3. ... ... $\dfrac{3ay^4}{\sqrt[3]{2ax^4}}$, . $= \dfrac{3y^4\sqrt[3]{4a^2x^2}}{2x^2}$.

4. ... ... $\dfrac{1}{\sqrt[r]{x^{2m}y^{3n}}}$, . $= \dfrac{\sqrt[r]{(x^{2m}y^{3n})^{r-1}}}{x^{2m}y^{3n}}$.

When the quantity under the radical sign, instead of being a single quantity as in the first case, is a single compound integral quantity as $(a - x)$, or $(x^2 + y^2)^2$, the same rule applies; for a single quantity as z may be assumed for the compound quantity under the parentheses; thus, let $(a - x) = z$, or $(x^2 + y^2)^2 = z^2$, and then apply the rule, and afterwards substitute for z its value. So when the quantity under the radical sign is the product of two or more compound integral quantities, a single letter may be assumed for each of them; and after applying the rule to these new quantities, their values can then be substituted for them.

This problem is of great utility in a practical point of view; for when the value of a fraction, such as $\dfrac{1}{\sqrt{2}}$, is to be found, the calculation is more tedious in its present form than when the denominator is rationalised, or reduced to the form $\dfrac{\sqrt{2}}{2}$ or $\dfrac{1}{2}\sqrt{2}$. For in the former case, $\sqrt{2}$ is to be found to as many decimal places as the degree of accuracy of the result requires, and then 1 must be divided by this approximate value of an interminate number; whereas in the latter case, after finding $\sqrt{2}$ to the required degree of accuracy, it is then merely to be divided by 2. The same remarks apply to algebraic expressions with irrational denominators; for when particular values are assigned to the

letters, the resulting expression will be of the fórm $\frac{1}{\sqrt[n]{a^m}}$, where a, m, and n, are numbers. There is one case of exception; that is, when there is a similar radical in the numerator, as $\frac{3\sqrt{8}}{2\sqrt{3}}$ or $\frac{3}{2}\sqrt{\frac{8}{3}}$ $= \frac{3}{2}\sqrt{2{\cdot}666'}$; or $\frac{a}{b}\sqrt[r]{\frac{x}{y}}$, in which case the value of $\frac{x}{y}$ is to be first found in numbers and then the given root of this number is to be extracted.

CASE IV.—TO REDUCE SURDS OF DIFFERENT DENOMINATIONS TO EQUIVALENT SURDS OF THE SAME DENOMINATION.

206. RULE. Reduce their exponents expressed under fractional forms to equivalent fractions having a common denominator.

EXAMPLES.

1. Reduce $\sqrt{a}$ and $\sqrt[3]{c}$ to surds of the same denomination.

The exponents are $\frac{1}{2}$ and $\frac{1}{3}$, and these being reduced to the same denominator, give $\frac{3}{6}$ and $\frac{2}{6}$;

hence $\sqrt{a} = a^{\frac{1}{2}} = a^{\frac{3}{6}}$, and $\sqrt[3]{c} = c^{\frac{1}{3}} = c^{\frac{2}{6}}$,

and the quantities are $a^{\frac{3}{6}}$ and $c^{\frac{2}{6}}$, or $\sqrt[6]{a^3}$ and $\sqrt[6]{c^2}$.

2. Reduce $\sqrt[3]{(a-x)^2}$ and $\sqrt{(a+x)^3}$ to a common radical sign.

$\frac{2}{3}$ and $\frac{3}{2}$ are equal to $\frac{4}{6}$ and $\frac{9}{6}$; hence the quantities are $(a-x)^{\frac{4}{6}}$ and $(a+x)^{\frac{9}{6}}$, or $\sqrt[6]{(a-x)^4}$ and $\sqrt[6]{(a+x)^9}$.

207. The rule is founded on this principle—*that the fractional exponent of a radical may be changed into any equivalent fraction without altering the value of the surd.*

For let $\sqrt[n]{a^m}$ be the given surd; then $\sqrt[n]{a^m} = a^{\frac{m}{n}}$; let $a^{\frac{m}{n}} = c$, then $\sqrt[n]{a^m} = c$, and hence $a^m = c^n$; therefore $(a^m)^r = (c^n)^r$, or $a^{mr} = c^{nr}$; and therefore $\sqrt[nr]{a^{mr}} = c$, that is, $a^{\frac{m}{n}} = a^{\frac{mr}{nr}}$.

EXERCISES.

1. Reduce $\sqrt{2}$ and $\sqrt[3]{3}$ to similar radicals, . $= 2^{\frac{3}{6}}$ and $3^{\frac{2}{6}}$.

2. ... $a^{\frac{1}{2}}$, $x^{\frac{1}{4}}$, and $y^{\frac{1}{3}}$, to similar surds, . $= a^{\frac{6}{12}}$, $x^{\frac{3}{12}}$, and $y^{\frac{4}{12}}$.

3. ... $2\sqrt{(a^2 - x^2)}$ and $3\sqrt[3]{(a^2 + x^2)}$ to the same radical sign, $= 2(a^2 - x^2)^{\frac{3}{6}}$ and $3(a^2 + x^2)^{\frac{2}{6}}$.

4. ... $\sqrt[n]{(a + x)^m}$ and $\sqrt[q]{(a - x)^p}$ to similar surds, $= (a + x)^{\frac{mq}{nq}}$ and $(a - x)^{\frac{np}{nq}}$.

CASE V.—ADDITION OF IRRATIONAL QUANTITIES.

208. Rule. Reduce the radicals to their simplest form, and if the surds be similar, add their coefficients, and prefix their sum to the radical part; but when the surds are different, their addition can only be represented by writing them in order, and connecting them by their proper signs.

EXAMPLES.

1. Add $2\sqrt{54}$ and $4\sqrt{24}$.

$$2\sqrt{54} = 2\sqrt{9}\sqrt{6} = 2 \times 3\sqrt{6} = 6\sqrt{6},$$

and $$4\sqrt{24} = 4\sqrt{4}\sqrt{6} = 4 \times 2\sqrt{6} = 8\sqrt{6};$$

therefore $2\sqrt{54} + 4\sqrt{24} = 6\sqrt{6} + 8\sqrt{6} = (6+8)\sqrt{6} = 14\sqrt{6}$.

2. Add $2\sqrt[3]{250}$ and $3\sqrt[3]{54}$.

$$2\sqrt[3]{250} = 2\sqrt[3]{125}\sqrt[3]{2} = 2 \times 5\sqrt[3]{2} = 10\sqrt[3]{2},$$

and $$3\sqrt[3]{54} = 3\sqrt[3]{27}\sqrt[3]{2} = 3 \times 3\sqrt[3]{2} = 9\sqrt[3]{2};$$

hence the sum $= 19\sqrt[3]{2}$.

3. Add $2\sqrt{\frac{3}{2}}$ and $3\sqrt{\frac{32}{27}}$.

$$2\sqrt{\frac{3}{2}} = 2\sqrt{\frac{6}{4}} = 2\frac{\sqrt{6}}{2} = \sqrt{6},$$

and $$3\sqrt{\frac{32}{27}} = 3\sqrt{\frac{96}{81}} = 3\frac{\sqrt{16}\sqrt{6}}{\sqrt{81}} = \frac{3 \times 4}{9}\sqrt{6} = \frac{4}{3}\sqrt{6};$$

and the sum $= \left(1 + \frac{4}{3}\right)\sqrt{6} = \frac{7}{3}\sqrt{6}$.

4. Add $2a\sqrt{\frac{x^3}{y}}$ and $3b\sqrt{\frac{x^5}{y^3}}$.

$$2a\sqrt{\frac{x^3}{y}} = 2a\sqrt{\frac{x^2xy}{y^2}} = \frac{2ax}{y}\sqrt{xy},$$

and $$3b\sqrt{\frac{x^5}{y^3}} = 3b\sqrt{\frac{x^4xy}{y^4}} = \frac{3bx^2}{y^2}\sqrt{xy};$$

hence the sum $= \left(\frac{2ax}{y} + \frac{3bx^2}{y^2}\right)\sqrt{xy} = \frac{x}{y^2}(2ay + 3bx)\sqrt{xy}$.

5. Add $3\sqrt{24}$, $2\sqrt{54}$, and $3\sqrt[3]{a^4x^5}$.

$$3\sqrt{24} = 3\sqrt{4}\sqrt{6} = 3 \times 2\sqrt{6} = 6\sqrt{6},$$

$$2\sqrt{54} = 2\sqrt{9}\sqrt{6} = 2 \times 3\sqrt{6} = 6\sqrt{6},$$

and $$3\sqrt[3]{a^4x^5} = 3\sqrt[3]{a^3x^3}\sqrt[3]{ax^2} = 3ax\sqrt[3]{ax^2};$$

and the sum $= 12\sqrt{6} + 3ax\sqrt[3]{ax^2}$.

EXERCISES.

1. Find the sum of $\sqrt{18}$ and $\sqrt{32}$, ... $= 7\sqrt{2}$.
2. ... ... $2\sqrt{54}$ and $3\sqrt{294}$, . . $= 27\sqrt{6}$.
3. ... ... $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$, ... $= \frac{19}{30}\sqrt{6}$.
4. ... ... $\sqrt[3]{80}$ and $\sqrt[3]{270}$, . . . $= 5\sqrt[3]{10}$.
5. ... ... $\sqrt{27a^4x^3}$ and $\sqrt{12a^2x^5}$, $= ax(3a + 2x)\sqrt{3x}$.
6. ... ... $4a\sqrt[3]{\frac{x^5}{y^4}}$ and $3cx\sqrt[3]{\frac{x^2}{y}}$, $= \frac{x}{y^2}(4a + 3cy)\sqrt[3]{x^2y^2}$.
7. ... ... $3a\sqrt{x^3y}$ and $5c\sqrt{xz}$, $= 3ax\sqrt{xy} + 5c\sqrt{xz}$.
8. ... ... $\frac{1}{2}\sqrt{ax}$, $3\sqrt[3]{a^4x}$, and $-2\sqrt{cx^3}$,

$$= \frac{1}{2}\sqrt{ax} + 3a\sqrt[3]{ax} - 2x\sqrt{cx}.$$

CASE VI.—SUBTRACTION OF IRRATIONAL QUANTITIES.

209. Rule. Change the sign of the subtrahend, and proceed as in addition of surds.

EXAMPLES.

1. From $12\sqrt{20}$ subtract $3\sqrt{45}$.

$$12\sqrt{20} = 12\sqrt{4}\sqrt{5} = 12 \times 2\sqrt{5} = 24\sqrt{5}$$
$$-3\sqrt{45} = -3\sqrt{9}\sqrt{5} = -3 \times 3\sqrt{5} = -9\sqrt{5};$$

hence the difference $= (24 - 9)\sqrt{5} = 15\sqrt{5}$.

2. Subtract $4\sqrt{\frac{6}{5}}$ from $6\sqrt{\frac{40}{27}}$.

$$4\sqrt{\frac{6}{5}} = 4\sqrt{\frac{30}{25}} = \frac{4}{5}\sqrt{30},$$

and $$6\sqrt{\frac{40}{27}} = 6\sqrt{\frac{120}{81}} = \frac{6}{9}\sqrt{4}\sqrt{30} = \frac{4}{3}\sqrt{30};$$

hence the difference $= \left(\frac{4}{3} - \frac{4}{5}\right)\sqrt{30} = \frac{8}{15}\sqrt{30}$.

3. From $3a^2\sqrt[3]{a^5x^2}$ subtract $2x^2\sqrt[3]{a^2x^5}$.

$$3a^2\sqrt[3]{a^5x^2} = 3a^2\sqrt[3]{a^3}\sqrt[3]{a^2x^2} = 3a^3\sqrt[3]{a^2x^2},$$

and $$2x^2\sqrt[3]{a^2x^5} = 2x^2\sqrt[3]{x^3}\sqrt[3]{a^2x^2} = 2x^3\sqrt[3]{a^2x^2};$$

hence the difference is $=(3a^3-2x^3)\sqrt[3]{a^2x^2}$.

4. From $5a\sqrt{\frac{x^3}{y^3}}$ subtract $3b\sqrt{\frac{x^5}{y^5}}$.

$$5a\sqrt{\frac{x^3}{y^3}} = 5a\sqrt{\frac{x^2xy}{y^4}} = \frac{5ax}{y^2}\sqrt{xy},$$

and $$3b\sqrt{\frac{x^5}{y^5}} = 3b\sqrt{\frac{x^4xy}{y^6}} = \frac{3bx^2}{y^3}\sqrt{xy};$$

and the difference $= \left(\frac{5ax}{y^2} - \frac{3bx^2}{y^3}\right)\sqrt{xy} = \frac{x}{y^3}(5ay - 3bx)\sqrt{xy}$.

5. From $5\sqrt[3]{a^4x^2}$ subtract $2\sqrt{a^5x^3}$.

$$5\sqrt[3]{a^4x^2} = 5\sqrt[3]{a^3}\sqrt[3]{ax^2} = 5a\sqrt[3]{ax^2},$$

and $$2\sqrt{a^5x^3} = 2\sqrt{a^4x^2}\sqrt{ax} = 2a^2x\sqrt{ax};$$

hence the difference $= 5a\sqrt[3]{ax^2} - 2a^2x\sqrt{ax}$.

EXERCISES.

1. From $5\sqrt{72}$ subtract $2\sqrt{32}$, ... $= 22\sqrt{2}$.

2. ... $2\sqrt{\frac{125}{12}}$ ... $4\sqrt{\frac{3}{5}}$, ... $= \frac{13}{15}\sqrt{15}$.

3. ... $3ax^2\sqrt{ax^3}$ ... $2a^2x\sqrt{a^3x}$, $= ax(3x^2 - 2a^2)\sqrt{ax}$.

4. ... $2c\sqrt{\frac{x}{y^3z}}$ ... $3a\sqrt{\frac{x^3}{yz^3}}$, . $= \frac{1}{y^2z^2}(2cz - 3axy)\sqrt{xyz}$.

5. ... $5\sqrt{(a-x)^3}$ ... $2a\sqrt{(a-x)}$, $= (3a-5x)\sqrt{(a-x)}$.

6. ... $\frac{3}{4}y\sqrt{x^3z}$ ... $\frac{1}{3}ax\sqrt[3]{x^4z}$, . $= \frac{3}{4}xy\sqrt{xz} - \frac{1}{3}ax^2\sqrt[3]{xz}$.

CASE VII.—TO MULTIPLY IRRATIONAL QUANTITIES.

210. Rule. Simplify the radicals, and, if necessary, reduce them to the same denomination; then multiply together the quantities that are under the same radical sign; and if they have coefficients, prefix the product of these.

When the quantities under the radical signs are powers of the same literal part, their product may be found by adding together their fractional exponents.

The multiplication may also be performed without previously simplifying the radicals.

EXAMPLES.

1. Multiply $3\sqrt{ax}$ by $2\sqrt{xy}$.

$$3\sqrt{ax} \times 2\sqrt{xy} = 6\sqrt{ax}\sqrt{xy} = 6\sqrt{ax^2y} = 6x\sqrt{ay}.$$

2. Multiply $2\sqrt{\frac{3}{5}}$ and $3\sqrt{\frac{5}{8}}$.

$$2\sqrt{\frac{3}{5}} \times 3\sqrt{\frac{5}{8}} = 2 \times 3\sqrt{\frac{3 \times 5}{5 \times 8}} = 6\sqrt{\frac{3}{8}} = 6\sqrt{\frac{6}{16}} = \frac{3}{2}\sqrt{6}.$$

3. Multiply $2a\sqrt{\frac{xy^3}{5z}}$ by $3x\sqrt{\frac{3z^3}{8x^3y}}$.

$$2a\sqrt{\frac{xy^3}{5z}} \times 3x\sqrt{\frac{3z^3}{8x^3y}} = 2a \times 3x\sqrt{\frac{xy^3 \times 3z^3}{5z \times 8x^3y}} = 6ax\sqrt{\frac{3xy^3z^3}{40x^3yz}} =$$

$$6ax\sqrt{\frac{3y^2z^2}{40x^2}} = \frac{6axyz}{x}\sqrt{\frac{3}{40}} = 6ayz\sqrt{\frac{30}{400}} = \frac{6ayz}{20}\sqrt{30} = \frac{3}{10}ayz\sqrt{30}.$$

4. Multiply $2\sqrt{(a-x)}$, $3ax\sqrt{(a+x)}$, and $5\sqrt{ax^3}$.

$$2\sqrt{(a-x)} \times 3ax\sqrt{(a+x)} \times 5\sqrt{ax^3} =$$
$$30ax\sqrt{\{ax^3(a-x)(a+x)\}} = 30ax^2\sqrt{\{ax(a^2-x^2)\}}.$$

5. Multiply $3a\sqrt{x^3}$ by $2x\sqrt[3]{a^2}$.

$3a\sqrt{x^3} \times 2x\sqrt[3]{a^2} = 6ax^2\sqrt{x}\sqrt[3]{a^2} = 6ax^2x^{\frac{1}{2}}a^{\frac{2}{3}} = 6ax^2x^{\frac{3}{6}}a^{\frac{4}{6}} = 6ax^2\sqrt[6]{a^4x^3}$.

6. Multiply $3a\sqrt{x^3}$ and $5b\sqrt[3]{x^2}$.

$3a\sqrt{x^3} \times 5b\sqrt[3]{x^2} = 15abx^{\frac{3}{2}}x^{\frac{2}{3}} = 15abx^{\frac{3}{2}+\frac{2}{3}} = 15abx^{\frac{13}{6}} = 15ab\sqrt[6]{x^{13}} = 15ab\sqrt[6]{x^{12}}\sqrt[6]{x} = 15abx^2\sqrt[6]{x}$.

7. Multiply $3\sqrt{(a-x)}$ and $4\sqrt[3]{(a-x)}$.

$3\sqrt{(a-x)} \times 4\sqrt[3]{(a-x)} = 12(a-x)^{\frac{1}{2}}(a-x)^{\frac{1}{3}} = 12(a-x)^{\frac{1}{2}+\frac{1}{3}} = 12(a-x)^{\frac{5}{6}} = 12\sqrt[6]{(a-x)^5}$.

The reason of the first part of the rule is evident from (177 and 186); and the truth of the latter part of it is evident from the following example:—

$$\sqrt[n]{a^m} \cdot \sqrt[q]{a^p} = a^{\frac{m}{n}} \cdot a^{\frac{p}{q}},$$

but $a^{\frac{m}{n}} = a^{\frac{mq}{nq}}$, and $a^{\frac{p}{q}} = a^{\frac{np}{nq}}$ (207); and hence the product is $= a^{\frac{mq}{nq}} \cdot a^{\frac{np}{nq}} = \sqrt[nq]{a^{mq}}\sqrt[nq]{a^{np}} = \sqrt[nq]{a^{mq}a^{np}} = \sqrt[nq]{a^{mq+np}}$, and this is $= a^{\frac{mq+np}{nq}}$, where the exponent is the sum of the given exponents.

EXERCISES.

1. Multiply $3\sqrt{ax^3}$, $2\sqrt{5}$, and $3\sqrt{x}$, . . . $= 18x^2\sqrt{5a}$.
2. ... $2\sqrt{\frac{3}{4}}$ and $5\sqrt{\frac{8}{9}}$, ... $= \frac{10}{3}\sqrt{6}$.
3. ... $2\sqrt{\frac{ax}{y^3}}$ and $3\sqrt{\frac{a^3y^5}{x^3}}$, ... $= \frac{6a^2y}{x}$.
4. ... $3a\sqrt[3]{(a-x)}$ and $5x\sqrt[3]{(a-x)^2}$, $= 15ax(a-x)$.
5. ... $2a\sqrt{x}$ and $3x\sqrt[3]{a}$, ... $= 6ax\sqrt[6]{a^2x^3}$.
6. ... $5a\sqrt{(a-x)}$, $2\sqrt[3]{ax}$, and $3\sqrt[4]{x^5}$,
$= 30ax\sqrt[12]{\{a^4x^7(a-x)^6\}}$.

7. Multiply $5\sqrt{x^5}$ and $8\sqrt[3]{x^4}$, ... $= 40x^3\sqrt[6]{x^5}$.

8. ... $3a\sqrt{(x^2 - y^2)}$ and $5b\sqrt[3]{(x^2 - y^2)}$, $= 15ab\sqrt[6]{(x^2 - y^2)^5}$.

9. ... $\sqrt[m]{(a - x)}$ by $\sqrt[n]{(a + x)}$, $= \{(a + x)^m(a - x)^n\}^{\frac{1}{mn}}$.

CASE VIII.—DIVISION OF IRRATIONAL QUANTITIES.

211. Rule. Simplify the radicals, and reduce them to the same denomination if necessary; then to the quotient of the radical parts, placed under the common radical sign, prefix that of their coefficients.

When the literal part of the quantities under the radical signs is the same, their quotient may be found by taking the difference of their fractional exponents.

The division may be performed without previously simplifying the radicals when similar; but they ought first to be simplified when they are dissimilar.

EXAMPLES.

1. Divide $5\sqrt{8}$ by $3\sqrt{27}$.

$$5\sqrt{8} \div 3\sqrt{27} = \frac{5}{3}\sqrt{\frac{8}{27}} = \frac{5}{3}\sqrt{\frac{24}{81}} = \frac{5\sqrt{4}\sqrt{6}}{3\sqrt{81}} = \frac{5}{3} \times \frac{2}{9}\sqrt{6} = \frac{10}{27}\sqrt{6}.$$

2. Divide $4\sqrt{a^3x}$ by $2\sqrt{ax^3}$.

$4\sqrt{a^3x} = 4a\sqrt{ax}$, and $2\sqrt{ax^3} = 2x\sqrt{ax}$, and $4a\sqrt{ax} \div 2x\sqrt{ax} = \frac{4a}{2x}\sqrt{\frac{ax}{ax}} = \frac{2a}{x}$.

3. Divide $5a\sqrt{\frac{x^3}{y}}$ by $4a\sqrt{\frac{x}{y^3}}$.

$$5a\sqrt{\frac{x^3}{y}} = 5ax\sqrt{\frac{xy}{y^2}} = \frac{5ax}{y}\sqrt{xy}, \text{ and } 4a\sqrt{\frac{x}{y^3}} = \frac{4a}{y^2}\sqrt{xy},$$

$$\text{and } \frac{5ax}{y}\sqrt{xy} \div \frac{4a}{y^2}\sqrt{xy} = \frac{5ax}{y} \times \frac{y^2}{4a}\sqrt{\frac{xy}{xy}} = \frac{5xy}{4}.$$

Or without previously simplifying,

$$5a\sqrt{\frac{x^3}{y}} \div 4a\sqrt{\frac{x}{y^3}} = \frac{5a}{4a}\sqrt{\frac{x^3 \times y^3}{y \times x}} = \frac{5a}{4a}\sqrt{x^2y^2} = \frac{5xy}{4}.$$

4. Divide $3a\sqrt{a^5x}$ by $2x\sqrt[3]{a^2x^5}$.

$3a\sqrt{a^5x} = 3a^3\sqrt{ax}$, and $2x\sqrt[3]{a^2x^5} = 2x^2\sqrt[3]{a^2x^2}$, and $\dfrac{3a^3\sqrt{ax}}{2x^2\sqrt[3]{a^2x^2}} =$

$$\frac{3a^3(ax)^{\frac{1}{2}}}{2x^2(a^2x^2)^{\frac{1}{3}}} = \frac{3a^3(ax)^{\frac{3}{6}}}{2x^2(a^2x^2)^{\frac{2}{6}}} = \frac{3a^3}{2x^2}\sqrt[6]{\frac{a^3x^3}{a^4x^4}} = \frac{3a^3}{2x^2}\sqrt[6]{\frac{a^5x^5}{a^6x^6}} = \frac{3a^3}{2x^2}\cdot\frac{\sqrt[6]{a^5x^5}}{ax} =$$

$$\frac{3a^2}{2x^3}\sqrt[6]{a^5x^5}.$$

5. Divide $2\sqrt{(a-x)^3}$ by $5\sqrt[3]{(a-x)^2}$.

$$\frac{2\sqrt{(a-x)^3}}{5\sqrt[3]{(a-x)^2}} = \frac{2(a-x)^{\frac{3}{2}}}{5(a-x)^{\frac{2}{3}}} = \frac{2}{5}(a-x)^{\frac{3}{2}-\frac{2}{3}} = \frac{2}{5}(a-x)^{\frac{5}{6}} = \frac{2}{5}\sqrt[6]{(a-x)^5}.$$

The first part of the rule is evident from (187); and the second part may be shewn thus—

$$a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{mq}{nq}} \div a^{\frac{np}{nq}} = \frac{\sqrt[nq]{a^{mq}}}{\sqrt[nq]{a^{np}}} = \sqrt[nq]{\frac{a^{mq}}{a^{np}}}\,(187) = \sqrt[nq]{a^{mq-np}}, \text{ or } a^{\frac{mq-np}{nq}}$$

where the exponent is $=$ the difference between the given exponents $\dfrac{m}{n}$ and $\dfrac{p}{q}$.

EXERCISES.

1. Divide $3\sqrt{24}$ by $5\sqrt{6}$, ... $= \dfrac{6}{5}$

2. ... $2a\sqrt{a^3x}$ by $5x\sqrt{ax^3}$, ... $= \dfrac{2a^2}{5x^2}$

3. ... $5\sqrt{\dfrac{3}{5}}$ by $3\sqrt{\dfrac{27}{20}}$, ... $= \dfrac{10}{9}$

4. ... $2a\sqrt{\dfrac{x^3z^5}{cy^3}}$ by $4x\sqrt{\dfrac{c^3xy^5}{2z^4}}$, . . . $= \dfrac{az^4}{2c^2y^4}\sqrt{2z}$

5. ... $2\sqrt{5}$ by $3\sqrt[3]{4}$, ... $= \dfrac{1}{3}\sqrt[6]{500}$

6. ... $5a\sqrt{a^3x}$ by $10x\sqrt[3]{a^2x^5}$, . . . $= \dfrac{a}{2x^3}\sqrt{a^5x^5}$

7. ... $5\sqrt{x^3}$ by $4\sqrt[3]{x^5}$, ... $= \dfrac{5}{4x}\sqrt[6]{x^5}$

8. ... $3a\sqrt{(x^2-y^2)^3}$ by $4x\sqrt[3]{(x^2-y^2)}$, $= \dfrac{3a(x^2-y^2)}{4x}\sqrt[6]{(x^2-y^2)}$

CASE IX.—INVOLUTION OF IRRATIONAL QUANTITIES.

212. RULE. To involve a surd, multiply its fractional exponent by the exponent of the required power; or multiply the exponent of the quantity under the radical sign by the exponent of the required power, and place the result under the radical sign; then simplify the result, if possible, and prefix the same power of the coefficient.

The required power is often more easily obtained if the radical is not simplified previously to applying the rule.

EXAMPLES.

1. Find the square of $3a\sqrt[3]{x^2}$.

$$(3a\sqrt[3]{x^2})^2 = 9a^2(x^{\frac{2}{3}})^2 = 9a^2x^{\frac{4}{3}} = 9a^2x\sqrt[3]{x},$$

or

$$(3a\sqrt[3]{x^2})^2 = 9a^2\sqrt[3]{x^4} = 9a^2x\sqrt[3]{x}.$$

2. Find the cube of $2\sqrt{3}$.

$$(2\sqrt{3})^3 = 8\sqrt{27} = 8\sqrt{9}\sqrt{3} = 8 \times 3\sqrt{3} = 24\sqrt{3}.$$

3. Find the cube of $2a\sqrt{\frac{xy^3}{z}}$.

$$\left(2a\sqrt{\frac{xy^3}{z}}\right)^3 = 8a^3\sqrt{\frac{x^3y^9}{z^3}} = 8a^3\sqrt{\frac{x^3y^9z}{z^4}} = \frac{8a^3xy^4}{z^2}\sqrt{xyz}.$$

4. Find the fourth power of $3\sqrt[3]{(a^2 - x^2)}$.

$$\{3\sqrt[3]{(a^2 - x^2)}\}^4 = 81\sqrt[3]{(a^2 - x^2)^4} = 81(a^2 - x^2) \times \sqrt[3]{(a^2 - x^2)}.$$

5. Find the cube of $2a\sqrt{\frac{x^3(a-y)}{a(x+y)^3}}$.

$$\left\{2a\sqrt{\frac{x^3(a-y)}{a(x+y)^3}}\right\}^3 = 8a^3\sqrt{\frac{x^9(a-y)^3}{a^3(x+y)^9}} = 8a^3\sqrt{\frac{ax^9(a-y)^3(x+y)}{a^4(x+y)^{10}}} =$$

$$\frac{8a^3x^4(a-y)}{a^2(x+y)^5}\sqrt{ax(a-y)} \times \sqrt{(x+y)} = \frac{8ax^4(a-y)}{(x+y)^5}\sqrt{ax(a-y)(x+y)}.$$

The truth of the rule may be proved thus:—

The rth power of $x^{\frac{m}{n}}$ is equal to the product arising from multiplying this quantity r times into itself, or $= x^{\frac{m}{n}} \cdot x^{\frac{m}{n}} \cdot x^{\frac{m}{n}}$ ..... the quantity $x^{\frac{m}{n}}$ being repeated r times as a factor. But this product

is found by adding the exponents; and as $\frac{m}{n}$ is to be repeated r times, the sum is $= \frac{mr}{n}$; that is, $(x^{\frac{m}{n}})^r = x^{\frac{mr}{n}}$ or $\sqrt[n]{x^{mr}}$.

EXERCISES.

1. Find the square of $2a\sqrt{x}$, ... $= 4a^2x$.
2. ... cube of $3ax\sqrt{xy}$, . . . $= 27a^3x^4y\sqrt{xy}$.
3. ... fourth power of $2x\sqrt[3]{\frac{ay^2}{4z^2}}$, . $= \frac{2ax^4y^2}{z^3}\sqrt[3]{2ay^2z}$.
4. ... cube of $3\sqrt{2}$, ... $= 54\sqrt{2}$.
5. ... fourth power of $3\sqrt[3]{\frac{5}{2}}$, . . . $= \frac{405}{4}\sqrt[3]{20}$.
6. ... square of $2a\sqrt[3]{(a-x)}$, . $= 4a^2\sqrt[3]{(a-x)^2}$.
7. ... fourth power of $2x\sqrt{\frac{x-y}{2z}}$, . $= \frac{4x^4(x-y)^2}{z^2}$.
8. ... cube of $3\sqrt{\frac{a+x}{a-x}}$, . $= \frac{27(a+x)}{(a-x)^2}\sqrt{(a^2-x^2)}$.

CASE X.—EVOLUTION OF IRRATIONAL QUANTITIES.

213. RULE. To extract a root of a radical, divide its fractional exponent by that of the root; or find the required root of the quantity under the radical sign, and place the result under this sign; then prefix the required root of the coefficient, and simplify the result if possible.

When the root of the coefficient is a radical, multiply it by the other radical part, according to the former rule for multiplying radicals (210).

EXAMPLES.

1. Find the square root of $9a^2\sqrt[3]{x^2}$.

$$\sqrt{(9a^2\sqrt[3]{x^2})} = (9a^2)^{\frac{1}{2}}(x^{\frac{2}{3}})^{\frac{1}{2}} = 3ax^{\frac{1}{3}} = 3a\sqrt[3]{x}.$$

2. Required the cube root of $8a^6\sqrt{y^3}$.

$$\sqrt[3]{(8a^6\sqrt{y^3})} = (8a^6)^{\frac{1}{3}}(y^{\frac{3}{2}})^{\frac{1}{3}} = 2a^2y^{\frac{1}{2}} = 2a^2\sqrt{y}.$$

3. Required the square root of $3a^3\sqrt[3]{x^2}$.

$$\sqrt{(3a^3\sqrt[3]{x^2})} = (3a^3)^{\frac{1}{2}}(x^{\frac{2}{3}})^{\frac{1}{2}} = 3^{\frac{1}{2}}a^{\frac{3}{2}}x^{\frac{1}{3}} = 3^{\frac{1}{2}}aa^{\frac{1}{2}}x^{\frac{1}{3}} = a\sqrt{3a}\,.\sqrt[3]{x},$$

or $$= a(3a)^{\frac{1}{2}}x^{\frac{1}{3}} = a(3a)^{\frac{3}{6}}x^{\frac{2}{6}} = a\sqrt[6]{27a^3x^2}.$$

4. Find the cube root of $27a^3\sqrt{\dfrac{x^3}{y^5}}$.

$$\sqrt[3]{\left(27a^3\sqrt{\frac{x^3}{y^5}}\right)} = 3a\left(\frac{x^3}{y^5}\right)^{\frac{1}{2}\times\frac{1}{3}} = 3a\sqrt[6]{\frac{x^3}{y^5}} = 3a\sqrt[6]{\frac{x^3y}{y^6}} = \frac{3a}{y}\sqrt[6]{x^3y}.$$

5. Required the square root of $8a^3x\sqrt[3]{(a-x)^2}$.

$$\sqrt{\{8a^3x\sqrt[3]{(a-x)^2}\}} = (8a^3x)^{\frac{1}{2}}(a-x)^{\frac{2}{3}\times\frac{1}{2}} = 2a(2ax)^{\frac{1}{2}}$$
$$\times(a-x)^{\frac{1}{3}} = 2a(2ax)^{\frac{3}{6}}(a-x)^{\frac{2}{6}} = 2a\sqrt[6]{\{8a^3x^3(a-x)^2\}}$$

The reason of the rule may be shewn thus—

By (212) $(a^{\frac{m}{nr}})^r = a^{\frac{mr}{nr}} = a^{\frac{m}{n}}$; hence, taking the rth root of these equals, $\sqrt[r]{a^{\frac{m}{n}}} = a^{\frac{m}{nr}}$.

EXERCISES.

1. Required the square root of $16x^4\sqrt[3]{x^4}$, . . . $= 4x^2x^{\frac{2}{3}}$.
2. ... ... cube root of $27a^3\sqrt{x^3y^6}$, . . $= 3ayx^{\frac{1}{2}}$.
3. ... ... square root of $10a^3x\sqrt[3]{y^2z^4}$, $= a(1000a^3x^3y^2z^4)^{\frac{1}{6}}$.
4. ... ... cube root of $8x^6\sqrt{\dfrac{3ay^3}{2z}}$, . $= \dfrac{x^2}{z}(96ay^3z^5)^{\frac{1}{6}}$.
5. ... ... fourth root of $81\sqrt[3]{\dfrac{2}{3}}$, . . . $= (2\times3^{11})^{\frac{1}{12}}$.
6. ... ... cube root of $8x^4\sqrt{(x^2-y^2)^3}$,
$= 2x\{x^2(x^2-y^2)^3\}^{\frac{1}{6}}$.

214. By means of the preceding rules on the calculation of surds, the operations of addition, subtraction, multiplication, and division, and also involution and evolution, may be performed on compound irrational quantities; that is, on compound quantities, all or some of whose terms are irrational.

EXAMPLES.

1. Add $3a\sqrt{x} + 4x$ and $5\sqrt{x} - 3ax + 2\sqrt{cz}$.

The sum is $= (3a+5)\sqrt{x} + (4-3a)x + 2\sqrt{cz}$.

2. From $8\sqrt{xy} - 2ax + 3\sqrt[3]{z^2}$ subtract $5\sqrt{y} - 2\sqrt[3]{z^2} + 5x$.

The difference $= 5\sqrt[3]{z^2} - (2a+5)x + 8\sqrt{xy} - 5\sqrt{y}$.

3. Multiply $3a\sqrt{x} - 2y + 2\sqrt[3]{z}$ by $2\sqrt{y} - x$.

$$\begin{array}{l}
3a\sqrt{x} - 2y + 2\sqrt[3]{z} \\
2\sqrt{y} - x \\
\hline
6a\sqrt{xy} - 4y\sqrt{y} + 4\sqrt[6]{y^3z^2} \\
- 3ax\sqrt{x} + 2xy - 2x\sqrt[3]{z} \\
\hline
\end{array}$$

As no two terms in the product are the same, addition will not alter its form.

4. Divide $x^2 - xz - y + z\sqrt{y}$ by $x - \sqrt{y}$.

$$\begin{array}{l}
x - \sqrt{y}\}x^2 - xz - y + z\sqrt{y}\{x + (\sqrt{y} - z) \\
\qquad x^2 - x\sqrt{y} \\
\hline
\qquad (\sqrt{y} - z)x - y + z\sqrt{y} \\
\qquad (\sqrt{y} - z)x - y + z\sqrt{y} \\
\hline
\end{array}$$

5. Find the square of $a^3 - 2x^{\frac{1}{2}}$.

$$\begin{array}{l}
a^3 - 2x^{\frac{1}{2}} \\
a^3 - 2x^{\frac{1}{2}} \\
\hline
a^6 - 2a^3x^{\frac{1}{2}} \\
\quad - 2a^3x^{\frac{1}{2}} + 4x \\
\hline
a^6 - 4a^3x^{\frac{1}{2}} + 4x \\
\hline
\end{array}$$

6. Extract the square root of $4x - 4ax^{\frac{1}{2}}y^{\frac{1}{3}} + a^2y^{\frac{2}{3}}$.

$$\begin{array}{l}
\{4x - 4ax^{\frac{1}{2}}y^{\frac{1}{3}} + a^2y^{\frac{2}{3}}\}^{\frac{1}{2}} = 2x^{\frac{1}{2}} - ay^{\frac{1}{3}} \\
4x \\
\hline
4x^{\frac{1}{2}} - ay^{\frac{1}{3}}\} - 4ax^{\frac{1}{2}}y^{\frac{1}{3}} + a^2y^{\frac{2}{3}} \\
\qquad - 4ax^{\frac{1}{2}}y^{\frac{1}{3}} + a^2y^{\frac{2}{3}} \\
\hline
\end{array}$$

This example is performed by the rule for extracting the square root of a compound quantity (189) with the preceding rules for irrational quantities. The rule for extracting the cube root (191) may be similarly applied.

EXERCISES.

1. Add $2x\sqrt{y} + 5\sqrt[3]{ax^2}$, $3z\sqrt{y} - 5$, and $8 + 2\sqrt[3]{ax^2}$,

$$= (2x + 3z)\sqrt{y} + 7\sqrt[3]{ax^2} + 3.$$

2. Subtract $3\sqrt{xy} - 5\sqrt[3]{z^2}$ from $8\sqrt{xy} - 12 + 8\sqrt[3]{z^2}$,

$$= 5\sqrt{xy} - 12 + 13\sqrt[3]{z^2}.$$

3. Multiply $2ay\sqrt{y} + 2\sqrt{(a-x)}$ by $3\sqrt{y} - 5\sqrt{(a-x)}$,

$$= 6ay^2 + 2(3 - 5ay)\sqrt{y(a-x)} - 10(a-x).$$

4. Divide $x^5 - (3a+2)x^3\sqrt{z} + 6axz$ by $x^2 - 3a\sqrt{z}$,

$$= x^3 - 2x\sqrt{z}.$$

5. Find the square of $ax^2 - 2x\sqrt{y}$, $= a^2x^4 - 4ax^3\sqrt{y} + 4x^2y$.

6. ... the square root of $9a^2x - 12ax\sqrt{xy} + 4x^2y$,

$$= 3a\sqrt{x} - 2x\sqrt{y}.$$

Having once established the existence of such quantities as surds (196), the following important theorems may be easily proved :—

215. THEOREM I. The square root of a quantity cannot be partly rational and partly a quadratic surd.

For, if possible, let $\sqrt{x} = a + \sqrt{y}$; then squaring these equals, $x = a^2 + y + 2a\sqrt{y}$; and hence

$$\sqrt{y} = \frac{x - a^2 - y}{2a};$$

that is, a surd is equal to a rational quantity, which is impossible.

It may be similarly shewn that $\sqrt[3]{x} = a + \sqrt{y}$ is an impossible equation, and generally, that any root of a quantity cannot consist of a rational and an irrational quantity.

216. THEOREM II. In an equation, the members of which consist partly of rational and quadratic irrational quantities, the rational parts of each member are equal, and also the irrational parts.

Let the equation be $a + \sqrt{x} = b + \sqrt{y}$, and if a be not $= b$, let $a = b + z$, where z may be a terminate number, but cannot be a surd (215); then

$$b + z + \sqrt{x} = b + \sqrt{y},$$

or

$$\sqrt{y} = z + \sqrt{x},$$

which is impossible (215); hence $a = b$, and therefore $x = y$.

In a similar manner it may be shewn generally, if $a + \sqrt[n]{x} = b + \sqrt[n]{y}$, that $a = b$, and $x = y$.

217. THEOREM III. When the denominator of a fraction is of the form $c^{\frac{1}{n}} \pm y^{\frac{1}{n}}$, it may be rationalised by multiplying the terms of the fraction by a proper multiplier.

The multiplier can be found thus :—

In the three cases of division (87) assume $a^n = c$ and $x^n = y$; and hence $a = c^{\frac{1}{n}}$, $a^2 = c^{\frac{2}{n}}$, $a^3 = c^{\frac{3}{n}}$ ... and $x = y^{\frac{1}{n}}$, $x^2 = y^{\frac{2}{n}}$, $x^3 = y^{\frac{3}{n}}$ ... and these cases become

1. When n is either an even or an odd number,

$$\frac{c - y}{c^{\frac{1}{n}} - y^{\frac{1}{n}}} = c^{\frac{n-1}{n}} + c^{\frac{n-2}{n}} y^{\frac{1}{n}} + c^{\frac{n-3}{n}} y^{\frac{2}{n}} + \ldots + y^{\frac{n-1}{n}}.$$

2. When n is an odd number,

$$\frac{c + y}{c^{\frac{1}{n}} + y^{\frac{1}{n}}} = c^{\frac{n-1}{n}} - c^{\frac{n-2}{n}} y^{\frac{1}{n}} + c^{\frac{n-3}{n}} y^{\frac{2}{n}} - \ldots + y^{\frac{n-1}{n}}.$$

3. When n is an even number,

$$\frac{c - y}{c^{\frac{1}{n}} + y^{\frac{1}{n}}} = c^{\frac{n-1}{n}} - c^{\frac{n-2}{n}} y^{\frac{1}{n}} + c^{\frac{n-3}{n}} y^{\frac{2}{n}} - \ldots - y^{\frac{n-1}{n}}.$$

It is evident from these expressions, that if the quantity to which any of these fractions is equal be multiplied by its denominator, the product will be the numerator, which is a rational quantity; and, therefore, if the terms of any fraction having the same denominator be multiplied by the same quantity, its denominator will be rationalised.

EXAMPLES.

1. Find a multiplier whose product by $2^{\frac{1}{2}} + 5^{\frac{1}{2}}$ shall be rational.

Here $c = 2$, $y = 5$, and $n = 2$; and hence by the third case the multiplier is $c^{\frac{n-1}{n}} - c^{\frac{n-2}{n}} y^{\frac{1}{n}}$, as only two terms can be taken, n being less than 3. These two terms become $2^{\frac{1}{2}} - 5^{\frac{1}{2}}$, for $c^{\frac{n-2}{n}} = c^{\frac{2-2}{2}} = c^0 = 1$, and

$$(2^{\frac{1}{2}} + 5^{\frac{1}{2}})(2^{\frac{1}{2}} - 5^{\frac{1}{2}}) = 2 - 5 = -3.$$

2. Find the factor that will rationalise the denominator of $\dfrac{1}{4^{\frac{1}{3}} - 3^{\frac{1}{3}}}$.

Here $c = 4$, $y = 3$, and $n = 3$; and hence by the first case the multiplier $= c^{\frac{n-1}{n}} + c^{\frac{n-2}{n}} y^{\frac{1}{n}} + c^{\frac{n-3}{n}} y^{\frac{2}{n}}$, taking only three terms, as

n is less than 4. These three terms become $4^{\frac{2}{3}} + (4 \times 3)^{\frac{1}{3}} + 3^{\frac{2}{3}} = 16^{\frac{1}{3}} + 12^{\frac{1}{3}} + 9^{\frac{1}{3}}$, and

$$(4^{\frac{1}{3}} - 3^{\frac{1}{3}})(16^{\frac{1}{3}} + 12^{\frac{1}{3}} + 9^{\frac{1}{3}}) = 1.$$

3. Rationalise the denominator of $\frac{x}{a^{\frac{1}{3}} - z^{\frac{1}{3}}}$.

Here $c = a$, $y = z$, and $n = 3$; and hence the multiplier in the first case becomes here $a^{\frac{2}{3}} + a^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}$, and

$$\frac{x(a^{\frac{2}{3}} + a^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}})}{(a^{\frac{1}{3}} - z^{\frac{1}{3}})(a^{\frac{2}{3}} + a^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}})} = \frac{x(a^{\frac{2}{3}} + a^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}})}{a - z}.$$

EXERCISES.

1. Find a multiplier whose product by $3^{\frac{1}{2}} - 2^{\frac{1}{2}}$ shall be rational,
$= 3^{\frac{1}{2}} + 2^{\frac{1}{2}}$.

2. Rationalise the denominator of the fraction $\frac{1}{5^{\frac{1}{3}} + 2^{\frac{1}{3}}}$,
$= \frac{1}{7}(25^{\frac{1}{3}} - 10^{\frac{1}{3}} + 4^{\frac{1}{3}})$.

3. ... ... of the fraction $\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$,
$= \frac{a + b - 2\sqrt{ab}}{a - b}$.

4. ... ... of the fraction $\frac{ax^2}{c^{\frac{1}{3}} - z^{\frac{1}{3}}}$,
$= \frac{ax^2(c^{\frac{2}{3}} + c^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}})}{c - z}$.

The rules for the multiplication, division, involution, and evolution of quantities, are generally applicable, whether their exponents be integral or fractional, positive or negative. The only cases for which this proposition requires to be proved, are those in which the exponents are negative, and either integral or fractional. This may be proved by considering that, conventionally, a^{-m} is the same as $\frac{1}{a^m}$, and $\frac{1}{a^{-m}}$ the same as a^m. So $\frac{a^m}{a^n} = a^{m-n}$, or $= \frac{1}{a^{n-m}}$; and similarly, $a^{-\frac{m}{n}}$ is the same as $\frac{1}{a^{\frac{m}{n}}}$, and $\frac{1}{a^{-\frac{m}{n}}}$ the same as $a^{\frac{m}{n}}$. By this means, any quantity with a negative exponent may be converted into one with a positive exponent; and the rules being then applied to the latter, the result may be changed into one with negative exponents.

CASE XI.—TO EXTRACT THE SQUARE ROOT OF A BINOMIAL SURD.

217*. The square root of a binomial, one of whose terms is rational, and the other a quadratic surd, may frequently be expressed by a binomial, one or both of whose terms are quadratic surds.

Let the given binomial be $a + \sqrt{b}$, and assume that

$$\sqrt{x} + \sqrt{y} = \sqrt{a + \sqrt{b}}$$

where the $\sqrt{x}$ and the $\sqrt{y}$ may be, either both irrational, or one rational and the other irrational; squaring both sides gives

$$x + y + 2\sqrt{xy} = a + \sqrt{b} \qquad \text{(A)}$$

Therefore (216) $x + y = a$, and $2\sqrt{xy} = \sqrt{b}$, from which it follows, by subtracting the second from the first, that

$$x + y - 2\sqrt{xy} = a - \sqrt{b} \qquad \text{(B)}$$

and extracting the square root, that

$$\sqrt{x} - \sqrt{y} = \sqrt{a - \sqrt{b}} \qquad \text{(C)}$$

Multiplying the original equation by (C) gives

$$x - y = \sqrt{a^2 - b} \qquad \text{(D)}$$

and half the sum of (A) and (B) is

$$x + y = a \qquad \text{(E)}$$

(E) + (D) gives

$$2x = a + \sqrt{a^2 - b},$$

$$\therefore \sqrt{x} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}};$$

and (E) − (D) gives

$$2y = a - \sqrt{a^2 - b},$$

$$\therefore \sqrt{y} = \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}.$$

Whence $\sqrt{x} \pm \sqrt{y} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$,

which is the general formula required.

EXAMPLE.

Extract the square root of $4 + 2\sqrt{3}$.

Here $a = 4$, $\sqrt{b} = 2\sqrt{3}$, and substituting these values in the general formula gives

$$\sqrt{4 + 2\sqrt{3}} = \sqrt{\frac{4 + \sqrt{16 - 12}}{2}} + \sqrt{\frac{4 - \sqrt{16 - 12}}{2}}$$

$$= \sqrt{3} - 1,$$

$$\therefore \sqrt{4 + 2\sqrt{3}} = 1 + \sqrt{3}.$$

In the same manner the square root of $4 - 2\sqrt{3}$ by the general formula is

$$\sqrt{4 - 2\sqrt{3}} = \sqrt{3} - 1.$$

When $a^2 - b$ is not a square, (D) is irrational, and consequently the general formula in this case becomes more complex than the original quantity, and the transformation is useless; hence the formula can only be applied so as to simplify the result when $\sqrt{a^2 - b}$ is a rational quantity.

EXERCISES.

1. Extract the square root of	$5 + 2\sqrt{6}$,	$=$	$\sqrt{2} + \sqrt{3}$.
2.	$6 \pm 2\sqrt{5}$,	$=$	$\sqrt{5} \pm 1$.
3.	$8 + 2\sqrt{7}$,	$=$	$\sqrt{7} + 1$.
4.	$37 \pm 2\sqrt{70}$,	$=$	$\sqrt{35} \pm \sqrt{2}$.
5.	$2\frac{1}{4} - \sqrt{5}$,	$=$	$\frac{1}{2}\sqrt{5} - 1$.
6.	$41 \pm 24\sqrt{2}$,	$=$	$4\sqrt{2} \pm 3$.

IMAGINARY QUANTITIES.

218. *Imaginary* quantities are those whose values cannot be assigned either by terminate or interminate numbers.

These quantities, which are also called *impossible* quantities, occur when it is required to find an even root of a negative quantity, which is unassignable in terms of any real quantities, either rational or irrational; for any even power of any real quantity, whether positive or negative, is positive, and therefore no even root of a negative quantity can be found.

Any root of a negative quantity is equal to the same root of that quantity taken positively, multiplied by the same root of -1.

For $-a = a(-1)$; therefore $\sqrt[m]{-a} = \sqrt[m]{a}\sqrt[m]{-1}$.

When m is an odd number, $\sqrt[m]{-1} = -1$ (186.)

When m is an even number, it will be equal to some power of 2, as 2^r, or to the product of some power of 2, as 2^r, by an odd number n.

Let $m = 2^r$, then the mth root of -1 is $= \sqrt[2^r]{-1}$, or equal to the square root taken r times in succession. Thus, if $r = 3$, $m = 2^3 = 8$, and $\sqrt[m]{-1} = \sqrt[8]{-1} = \sqrt{\{\sqrt{(\sqrt{-1})}\}}$, which is imaginary, because $\sqrt{-1}$ is so.

Let $m = 2^r n$, then $\sqrt[m]{-1} = \sqrt[2^r]{(\sqrt[n]{-1})}$; and as n is odd, $\sqrt[n]{-1} = -1$; therefore $\sqrt[m]{-1} = \sqrt[2^r]{-1}$; and, as in the preceding case, the imaginarity of this latter expression depends on $\sqrt{-1}$, as is evident by giving r any particular value, as 3, as is shewn above.

219. All imaginary quantities therefore depend on the single imaginary $\sqrt{-1}$. Also any quadratic imaginary is the product of a real quantity and $\sqrt{-1}$. For $\sqrt{-a} = \sqrt{a(-1)} = \sqrt{a}\sqrt{-1}$.

The product of two quadratic imaginaries leads to a paradoxical result. Thus, $\sqrt{-a}\,.\sqrt{-a}$ is the square of the square root of $-a$; and hence the result must be $-a$. But by the rule for multiplying radicals, $\sqrt{-a}\,.\,\sqrt{-a} = \sqrt{(-a)(-a)} = \sqrt{a^2}$, and this last quantity is usually expressed by $\pm a$; so that if the upper sign be taken, the result is $+a$, whereas the former

product was $-a$. This apparent contradiction is, however, easily reconciled. In the present case a^2 is the product $(-a)(-a)$, and its root is therefore $-a$; and being so, it cannot also be $+a$, for the double sign is properly used only when it is doubtful what the sign of the root is, and merely means that it is either positive or negative, but it does not imply that it has both these signs; and when the origin of the square is known in any case, as in the present, no ambiguity exists, so that the sign of the root being known, the other sign is excluded. Thus, when a^2 is the product $(-a)(-a)=(-a)^2$; then $\sqrt{a^2}=\sqrt{(-a)^2}=-a$; and when a^2 is the product $(+a)(+a)=(+a)^2$, $\sqrt{a^2}=\sqrt{(+a)^2}=+a$.

Since $\sqrt{-a}\,.\sqrt{-a}=-a$, therefore $\sqrt{-1}\sqrt{-1}=-1$.

The rules for addition and subtraction of imaginaries are the same as for surds.

CASE I.—MULTIPLICATION OF QUADRATIC IMAGINARIES.

220. Rule. The product of two quadratic imaginaries is found by taking the square root of their product, considering them as real, and prefixing the negative or positive sign, according as their signs are like or unlike.

For $(+\sqrt{-a})(+\sqrt{-b})=\sqrt{a}\sqrt{-1}\,.\sqrt{b}\sqrt{-1}=\sqrt{ab}\sqrt{(-1)^2}$
$$=-\sqrt{ab},$$

and $\quad(-\sqrt{-a})(-\sqrt{-b})=\sqrt{ab}\,.\sqrt{(-1)^2}=-\sqrt{ab}$;

also $\quad(-\sqrt{-a})(+\sqrt{-b})=-\sqrt{ab}\sqrt{(-1)^2}=+\sqrt{ab}$.

If the imaginaries have coefficients, the product of the former must be multiplied by that of the latter.

EXAMPLES.

1. Multiply $4\sqrt{-3}$ by $3\sqrt{-5}$.

$$4\sqrt{-3}\times 3\sqrt{-5}=-12\sqrt{15},$$

or $\quad 4\sqrt{3}\sqrt{-1}\times 3\sqrt{5}\sqrt{-1}=-12\sqrt{15}$.

2. Multiply $3+2\sqrt{-5}$ by $2-3\sqrt{-2}$.

$$\begin{array}{l}3+2\sqrt{-5}\\ 2-3\sqrt{-2}\\ \hline 6+4\sqrt{-5}\\ \qquad -9\sqrt{-2}+6\sqrt{10}\\ \hline 6+4\sqrt{-5}-9\sqrt{-2}+6\sqrt{10}\\ \hline\end{array}$$

3. Multiply $a - b\sqrt{-1}$ by $a + b\sqrt{-1}$.

$$(a - b\sqrt{-1})(a + b\sqrt{-1}) = a^2 + ab\sqrt{-1} - ab\sqrt{-1} + b^2$$
$$= a^2 + b^2.$$

EXERCISES.

1. Multiply $5\sqrt{-2}$ by $3\sqrt{-3}$, . . . $= -15\sqrt{6}$
2. ... $3\sqrt{-5}$ by $-2\sqrt{-3}$, . . . $= 6\sqrt{15}$
3. ... $3 + \sqrt{-2}$ by $2 - 3\sqrt{-1}$,
$$= 6 + 2\sqrt{-2} - 9\sqrt{-1} + 3\sqrt{2}$$
4. ... $a - b\sqrt{-c}$ by $a + b\sqrt{-c}$, . . $= a^2 + b^2c$
5. Find the square of $a - b\sqrt{-1}$, $= a^2 - 2ab\sqrt{-1} - b^2$
6. ... product of $a - 3\sqrt{-b}$ and $c - 2\sqrt{-d}$,
$$= ac - 2a\sqrt{-d} - 3c\sqrt{-b} - 6\sqrt{bd}$$

CASE II.—DIVISION OF QUADRATIC IMAGINARY QUANTITIES.

221. Rule. The quotient of two quadratic imaginaries is found by dividing them as if they were real, and prefixing the positive or negative sign to the quotient, according as their signs are like or unlike.

For $$\frac{+\sqrt{-a}}{+\sqrt{-b}} = \frac{+\sqrt{a}\sqrt{-1}}{+\sqrt{b}\sqrt{-1}} = +\sqrt{\frac{a}{b}},$$

so $$\frac{-\sqrt{-a}}{-\sqrt{-b}} = \frac{-\sqrt{a}\sqrt{-1}}{-\sqrt{b}\sqrt{-1}} = +\sqrt{\frac{a}{b}};$$

and $$\frac{+\sqrt{-a}}{-\sqrt{-b}} = \frac{+\sqrt{a}\sqrt{-1}}{-\sqrt{b}\sqrt{-1}} = -\sqrt{\frac{a}{b}}.$$

EXAMPLES.

1. Divide $8\sqrt{-12}$ by $2\sqrt{-3}$.

$$\frac{8\sqrt{-12}}{2\sqrt{-3}} = 4\sqrt{\frac{12}{3}} = 4\sqrt{4} = +4 \times 2 = +8.$$

2. Divide $-12\sqrt{-6}$ by $4\sqrt{-3}$.

$$\frac{-12\sqrt{-6}}{4\sqrt{-3}} = -3\sqrt{\frac{6}{3}} = -3\sqrt{2}.$$

3. ... $2+3\sqrt{-1}$ by $2-\sqrt{-1}$.

The multiplier that will make the denominator rational is $2+\sqrt{-1}$, which may be found from the general formula (217 *), by assuming $c^{\frac{1}{2}}=2$, and $y^{\frac{1}{2}}=\sqrt{-1}$; or it may be found more easily by considering that $(a+x)\times(a-x)=a^2-x^2$. Hence

$$\frac{2+3\sqrt{-1}}{2-\sqrt{-1}} = \frac{(2+3\sqrt{-1})(2+\sqrt{-1})}{(2-\sqrt{-1})(2+\sqrt{-1})} = \frac{1+8\sqrt{-1}}{4+1}$$

$$= \frac{1}{5}(1+8\sqrt{-1}), \text{ or } \frac{1}{5}+\frac{8}{5}\sqrt{-1}.$$

4. Divide $a+b\sqrt{-1}$ by $c+d\sqrt{-1}$.

The multiplier for rationalising the denominator in this example is evidently $c-d\sqrt{-1}$; hence

$$\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}} = \frac{(a+b\sqrt{-1})(c-d\sqrt{-1})}{(c+d\sqrt{-1})(c-d\sqrt{-1})} = \frac{ac+bd-(ad-bc)\sqrt{-1}}{c^2+d^2}.$$

EXERCISES.

1. Divide $8\sqrt{-18}$ by $2\sqrt{-2}$, . . . $= 4\sqrt{9}$ or 12.
2. ... $-15\sqrt{-6}$ by $3\sqrt{-3}$, . . . $= -5\sqrt{2}$.
3. ... $4-2\sqrt{-1}$ by $2-3\sqrt{-1}$, $= \frac{1}{13}(14+8\sqrt{-1})$.
4. ... $3-5\sqrt{-3}$ by $2-3\sqrt{-2}$,

$$= \frac{1}{22}(6-10\sqrt{-3}+9\sqrt{-2}+15\sqrt{6}).$$

5. ... $a\sqrt{-b}$ by $c\sqrt{-d}$, ... $= \frac{a}{cd}\sqrt{bd}$.
6. ... $2x+3y\sqrt{-1}$ by $3x-2y\sqrt{-1}$,

$$= \frac{6x^2-6y^2+13xy\sqrt{-1}}{9x^2+4y^2}.$$

EQUATIONS.

222. An *equation* is an expression stating the equality of two quantities, and generally containing at least one unknown quantity.

Thus, $x - 3 = 4$ is an equation which states the equality between $x - 3$ and 4; in which x is the unknown quantity. By the equation, it appears that if 3 be subtracted from x, the remainder is 4; and hence x must be 7. The equation $x = 5$ cannot be said to contain an unknown quantity, as the value of x is given; and so of the equations $x = 5-3$, $x = a$, or $x = b + c$; in which a, b, and c, are known quantities.

223. A quantity is *known* when its value in numbers is given; and when this value is not given, it is called an *unknown* quantity.

224. The last letters of the alphabet, x, y, z, are used to represent unknown quantities, and the known quantities are either numbers, or are denoted by the first letters of the alphabet, a, b, c, ...

225. The two parts of an equation on the opposite sides of the sign of equality, are called *members* or *sides* of the equation; that on the left is called the *first* member or side, and the other the *second* member or side. Members are composed of one or more terms.

226. The *solution* or method of *solving* an equation is the process for finding the value of the unknown quantity; and an equation is said to be *solved* when the unknown quantity stands alone on one side, and its value in known terms on the other side.

227. An equation consisting of literal quantities, when their numerical values are known, is called simply an *equality*.

Thus, if the numerical values of a, b, c, and d, be known, and be such that

$$a + b = c - d,$$

the expression is an equality. If, for example, $a = 2$, $b = 3$, $c = 8$, $d = 3$, then the expression becomes

$$2 + 3 = 8 - 3 \text{ or } 5 = 5.$$

228. A self-evident equality is called an *identity*, or a *verified equality*.

As $12 - 8 = 4$ or $3a - 2b - a = 2a - 5b + 3b$.

229. If, in an equation containing only one unknown quantity, a number be substituted for the unknown quantity, ... that the

equation is reduced to an identity, this number is called a *value* of the unknown quantity, or a *root* of the equation.

Thus, in the equation

$$x - 5 = 16 - 3 + 2,$$

if 20 be substituted for x, it becomes

$$20 - 5 = 16 - 3 + 2,$$

or

$$15 = 15.$$

This result is an identity; and hence 20 is the value of x.

230. An equation expresses some *relation* between the unknown and the known quantities. The value of the unknown quantity is said to *satisfy* or *fulfil* the conditions of the equation.

In the equation

$$x = 8 - 5$$

the relation between x and the known quantities is simply that x is equal to the difference between 8 and 5 — that is, 3; and this value of x being put in the place of x, reduces the equation to the identity $3 = 8 - 5$, or it satisfies the given equation.

231. To *verify* a value of the unknown quantity is to substitute this value for it in the given equation; if the result be an identity, the value is then *verified*, or proved to be correct.

232. If only one quantity in any proposed question be unknown, only one *relation* or *condition* must be given.

For example, if a number be required, such that being increased by 3, the sum shall be 12; then if x be that number, the equation

$$x + 3 = 12$$

expresses this relation or condition, and x is evidently $= 9$, for $9 + 3 = 12$. But it is unnecessary to give another relation such as that the number required being increased by 6, the sum is 15, as nothing would be deduced from this further than that the number required is 9. If, however, a second condition be added, so as to give a new value to the unknown quantity different from that given by the first equation, the question becomes *impossible*, as the conditions would be *inconsistent* were $x = 9$, and, at the same time, equal to some other number, while by the question only *one* number is sought. If, therefore, when only one unknown number is sought, a second condition be added, though it be consistent with the first, it is *unnecessary*, the first being *sufficient* for determining the required number; and if the second condition be *inconsistent* with the first, the question becomes *impossible*.

233. Equations are divided into *numerical* and *literal;* the former contain only numbers combined with the unknown quan-

tity or quantities, and the latter contain also literal quantities, whose numerical values are known, or supposed to be so.

Equations are also divided into *determinate* and *indeterminate;* the former term being applied when there are as many equations as unknown quantities, and the latter when there are more unknown quantities than equations.

234. Any power of a quantity, or any product of two or more quantities, or of their powers, is called a *dimension.* The order of the dimension depends on the number of times that the simple power of the quantity or quantities enters into the dimension, or on the sum of the exponents in the product. When the simple power is repeated twice, it is called the *second* dimension; when three times, the *third* dimension; and so on: and, generally, if it be repeated n times—that is, if the sum of the exponents be equal to n, it is called the nth dimension.

Thus, if x, y, z, be unknown quantities, x^3, x^2y, xyz, are each of the third dimension; x^5, xy^2z^2, x^3yz, x^4z, are of the fifth dimension, and so on.

235. Equations containing only one unknown quantity are divided into different classes, according to the highest power of the unknown quantity contained in them. An equation which contains only the first power of the unknown quantity, is called a *simple* equation; one in which the highest power is the second, is called a *quadratic;* when the highest power is the third, the equation is called a *cubic*, or an equation of the *third degree;* when the highest power is the fourth, an equation of the *fourth degree;* when the fifth, of the *fifth degree;* and, in general, when the highest power of the unknown quantity has the exponent n, it is called an equation of the *nth degree*, and sometimes it is called an equation of *n dimensions.*

236. Equations containing two or more unknown quantities are similarly classified. If the highest dimension of the unknown quantities be the first, second, or third, it is called a *simple*, *quadratic*, or *cubic* equation; if it contain the fourth dimension, it is called an equation of the *fourth degree* or *dimension;* and similarly for other dimensions.

237. The dimension of an equation is the same as that of any one of its terms which contains the highest dimension of the unknown quantity or quantities in the equation (40). It is understood, however, that the equation contains no fractional or negative exponents of the unknown quantity, or that it is cleared of radicals and of denominators containing unknown quantities.

SIMPLE EQUATIONS.

I.—EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

Equations must undergo some alterations in their form, in order to prepare them for solution. These preparatory transformations depend on the following axioms :—

238. Axioms. If equal quantities be added to the two members of an equation, or subtracted from them, or if the two members of an equation be multiplied by the same quantity, or divided by it, the results will still be equal, or the equation will still subsist.

From these axioms, the following *rules* for the process of solution are derived :—

239. Rule I. Any term may be transposed from one side of an equation to the other, by changing its sign.

EXAMPLES.

Transpose the known quantities to the second member, and the unknown to the first, in the following equations :—

1. Let $x - 4 = 6$.

Transposing the -4, we have $x = 6 + 4 = 10$.

2. Let $x + 8 = 4 - 3x$.

Transposing $+8$ and $-3x$, the result is

$$x + 3x = 4 - 8,$$

or $$4x = -4.$$

3. Let $x - 2a = -4x + 3b$.

$$x + 4x = 3b + 2a,$$

or $$5x = 2a + 3b.$$

240. The transposition of a negative quantity from one side of an equation to the other, is equivalent to *adding* that quantity to both sides; but the transposition of a positive quantity is equivalent to *subtracting* it from both sides. Thus, let

$$ax - b = cx - d;$$

by transposing cx, we have $ax - cx - b = -d$,

which is the same as subtracting cx from both sides. Again, by transposing $-b$, the equation becomes

$$ax - ax = b - d,$$

which is equivalent to adding b to both sides.

Or, by adding $(b - cx)$ to both sides, the equation becomes

$$ax - b + (b - cx) = cx - d + (b - cx),$$

or $$ax - cx = b - d;$$

for $-b + b$ destroy each other, and in like manner cx and $-cx$.

EXERCISES.

1. Let $5x - 3 = 2x + 5$, . . . $\therefore 3x = 8$.
2. ... $12 - 3x = 6 - 10x$, . . $\therefore 7x = -6$.
3. ... $5x - 6 - x = 15 + 2x - 3$, . $\therefore 2x = 18$.
4. ... $3x - 2a = 5c - 2x$, . . $\therefore 5x = 2a + 5c$.
5. ... $3a - 5x + 2c = 2x - 5c + 8a$, $\therefore -7x = 5a - 7c$.
6. ... $5x + 16 - 2x + 3a = 5a - x + 8$, $\therefore 4x = 2a - 8$.

241. As a corollary from the principle of transposition, we infer that the signs of *all* the terms of both sides of an equation may be changed, without destroying the equality of the two members.

For changing all the signs is the same as if all the terms of each member were transposed to the other side of the equation; or the effect is the same as if both members were multiplied by -1.

242. Rule II. An equation may be cleared of fractions, by multiplying both sides successively by the denominators; or by the least common multiple of the denominators.

The last method is always the best when the denominators are not all prime, as it gives an equation in more simple terms.

EXAMPLES.

1. Clear of fractions the equation $\frac{3}{4}x - \frac{2}{3}x = 11$.

Multiply by 4, and $3x - \frac{8}{3}x = 44$.

... by 3 ... $9x - 8x = 132$.

Or, more simply, multiply the equation at once by 12, and the result is

$$9x - 8x = 132.$$

2. Clear of fractions the equation $\frac{2}{3}x - \frac{3}{4} = \frac{5x}{8} + \frac{5}{6}$.

Multiplying by 24, we find $16x - 18 = 15x + 20$.

3. Clear of fractions $\dfrac{a}{a-x} - \dfrac{c}{a+x} = \dfrac{1}{a-x} + 3$.

Multiplying by $a^2 - x^2$, we find

$$a(a+x) - c(a-x) = a + x + 3(a^2 - x^2).$$

Rule II. depends on the axiom, that if both members of an equation be multiplied by the same quantity, the products are equal.

EXERCISES.

Clear the following equations of fractions :—

1. $\dfrac{x}{2} - 4 = \dfrac{x}{3} + 6$, . . . $\therefore\ 3x - 24 = 2x + 36$.

2. $\dfrac{3x}{8} - \dfrac{5}{6} = \dfrac{3}{4} + \dfrac{5x}{2}$, . . . $\therefore\ 9x - 20 = 18 + 60x$.

3. $\dfrac{x-4}{3} - \dfrac{2}{5} = \dfrac{6-x}{10} - 4$, $\therefore\ 10x - 40 - 12 = 18 - 3x - 120$.

4. $\dfrac{1}{x-2} - 3 = 5 - \dfrac{3}{x+2}$,

$$\therefore\ x + 2 - 3(x^2 - 4) = 5(x^2 - 4) - 3(x - 2).$$

5. $\dfrac{3-4x}{6} - \dfrac{3}{8} = \dfrac{5}{12} - \dfrac{3x+10}{9}$, $\therefore 36 - 48x - 27 = 30 - 24x - 80$.

6. $\dfrac{4}{a-x} - 8 = \dfrac{6}{x} - \dfrac{4}{a+x}$,

$$\therefore\ 4x(a+x) - 8x(a^2 - x^2) = 6(a^2 - x^2) - 4x(a - x).$$

7. $\dfrac{6}{x} - \dfrac{3x}{4} = \dfrac{5a}{3x} - \dfrac{2}{a+x}$,

$$\therefore\ 72(a+x) - 9x^2(a+x) = 20a(a+x) - 24x.$$

8. $\dfrac{a-x}{x} - \dfrac{3a}{b} = \dfrac{c}{a+x} - \dfrac{2b}{a}$,

$$\therefore\ ab(a^2 - x^2) - 3a^2x(a+x) = abcx - 2b^2x(a+x).$$

SOLUTION OF SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

243. Rule. If necessary, clear the equation of fractions; transpose to one side all the terms containing the unknown quantity, and the known quantities to the other; collect the terms on each side into one sum; then divide both sides by the coefficient of the unknown quantity: the result will be the value of this quantity.

EXAMPLES.

1. Given $3x - 8 = 24 - 5x$, to find the value of x.

Transposing, $3x + 5x = 24 + 8$.

Collecting, $8x = 32$.

Dividing by 8, $x = 4$.

In order to shew that this is the correct value of x, substitute 4 for x in the given equation, and it becomes

$$3 \times 4 - 8 = 24 - 5 \times 4,$$

or $$12 - 8 = 24 - 20;$$

and $$4 = 4:$$

which being an identity, proves the value to be correct, or *verifies* the equation.

2. Given $2x - 10 + x - 15 = 24 - 3x + 2x - 33$, to find the value of x.

In this example, by collecting the quantities in each member, we find

$$3x - 25 = -x - 9.$$

Transposing, $3x + x = 25 - 9$.

Collecting, $4x = 16$.

Dividing by 4, $x = 4$.

3. Given $4x - b = 2x - d$, to find the value of x in terms of b and d, the numerical values of which are supposed to be known.

Transposing $2x$ and $-b$, $4x - 2x = b - d$.

Collecting, $2x = b - d$.

Dividing by 2, $x = \frac{b - d}{2}$ or $\frac{1}{2}(b - d)$.

If numerical values be given to b and d, the corresponding value of x will be found.

Let $b = 5$ and $d = 3$; then $x = \frac{1}{2}(5 - 3) = \frac{1}{2} \times 2 = 1.$

... $b = 11$ and $d = 5$; then $x = \frac{11 - 5}{2} = \frac{6}{2} = 3.$

4. Let $ax - b = cx - d$, to find the value of x.

Transposing cx and $-b$, $ax - cx = b - d.$

Collecting, $(a - c)x = b - d.$

Dividing by $a - c$, $x = \dfrac{b - d}{a - c}.$

Suppose $b = 13$, $d = 7$, $a = 8$, $c = 5$,

then $x = \dfrac{b - d}{a - c} = \dfrac{13 - 7}{8 - 5} = \dfrac{6}{3} = 2.$

5. Let $\frac{x}{6} + 3 - \frac{x}{8} = 10 - \frac{x}{4}$, to find the value of x.

Multiplying by 24, we find $4x + 72 - 3x = 240 - 6x.$

Transposing, $4x - 3x + 6x = 240 - 72.$

Collecting, $7x = 168.$

Dividing by 7, $x = 24.$

6. Given $\frac{3}{4} - \frac{x - 2}{3} = \frac{5}{4} - \frac{x + 3}{4}$, to find the value of x.

Multiplying by 12, $9 - 4x + 8 = 15 - 3x - 9,$

or $-4x + 3x = 15 - 9 - 9 - 8;$

hence $-x = 15 - 26 = -11,$

or $x = 11.$

7. Let $\frac{x}{a} + b = \frac{x}{c} + d$, to find the value of x.

Multiplying by ac, we find $cx + abc = ax + acd,$

and $abc - acd = ax - cx,$

or $ac(b - d) = (a - c)x;$

hence $x = \dfrac{ac(b-d)}{a-c}.$

By giving a, b, c, and d, numerical values, the corresponding values of x may be found.

EXERCISES.

1. If $5x - 12 = 12 - 3x$, ... $x = 3$

2. ... $15 - 2x = 6x - 25$, ... $x = 5$

3. ... $5x - 10 - 2x = 40 + 4x - 56$, . . . $x - 6$

4. ... $\frac{x}{2} - 2 = 5 - \frac{x}{5}$, ... $x = 10.$

5. ... $\frac{x}{2} - \frac{x}{4} + 15 = \frac{x}{8} - \frac{x}{6} + 22$, . . . $x = 24.$

6. ... $\frac{x-3}{4} - 6 - \frac{x-1}{5} = \frac{x-5}{3} - 8$, . . $x = 11.$

7. ... $\frac{x+3}{4} - \frac{x-3}{5} = \frac{x-5}{2} - 2$, . . . $x = 13.$

8. ... $\frac{x+6}{4} - \frac{16-3x}{12} = \frac{25}{6}$, ... $x = 8.$

9. ... $\frac{x+1}{2} + \frac{x+2}{3} = 16 - \frac{5x+1}{4}$, . . $x = 7.$

10. ... $3(3x - 2) + 5(x - 6) = 18x - 4$, . . $x = -8.$

11. ... $\frac{x+1}{2} + \frac{x+2}{3} - \frac{5-x}{4} = 14$, . . . $x = 13.$

12. ... $\frac{4}{5}x - \frac{5}{4}x + 18 = \frac{1}{9}(4x + 1)$, ... $x = 20.$

13. ... $\frac{x}{a} - \frac{x-b}{c} = \frac{de}{ac}$, ... $x = \frac{ab - de}{a - c}.$

14. ... $\frac{x}{a-b} = \frac{x}{a+b} + 1$, ... $x = \frac{a^2 - b^2}{2b}.$

15. ... $\frac{x}{a} + c = \frac{x}{b} - d$, ... $x = \frac{ab(c+d)}{a-b}.$

16. ... $3ax - \frac{2b}{a} = \frac{3cx - bc + 2}{4c}$, . $x = \frac{2a + (8 - a)bc}{3ac(4a - 1)}.$

17. ... $\frac{(a-b)x}{a+b} + c = \frac{(a+b)x}{a-b} - d$, $x = \frac{(c+d)(a^2 - b^2)}{4ab}.$

18. ... $\frac{a}{bx} + \frac{b}{ax} = a^2 + b^2$, ... $x = \frac{1}{[illegible]}.$

244. When the unknown quantity enters into the terms of a proportion, an equation may be formed by making the product of the extremes equal to that of the means. Thus,

Let $m : x = a : b$, then $ax = bm$.

... $a - x : x = a : b$, then $ax = b(a - x)$.

I. — EQUATIONS THAT MAY BE SIMPLIFIED, AND WHICH CONTAIN THE UNKNOWN QUANTITY IN ALL THEIR TERMS, BUT IN ONLY TWO OF ITS POWERS.

245. Rule. Divide each term by the highest power of the unknown quantity, which is common to them all.

EXAMPLE.

1. Given $5x^2 - 3x = 16x - \frac{9x^2}{2}$, to find the value of x.

Dividing by x, we find $\quad 5x - 3 = 16 - \frac{9x}{2}$ 2

Multiplying by 2, $\quad 10x - 6 = 32 - 9x$

$$19x = 32 + 6 = 38$$

$$x = 2$$

EXERCISES.

1. If $3x^2 - 8x = 24x - 5x^2$, ... $x = 4$.
2. ... $5x^2 - 15x = 2x^2 + 6x$, ... $x = 7$.
3. ... $40x^2 - 6x^3 - 16x^2 = 120x^2 - 14x^3$, . . . $x = 12$.
4. ... $3ax^3 - 10ax^2 = 8ax^2 + ax^3$, ... $x = 9$.
5. ... $2x^3 - \frac{x^3}{2} + x^2 = 5x^3 - 2x^2$, ... $x = \frac{6}{7}$.
6. ... $x^2 + \frac{2x^2}{3} - \frac{x^2}{2} = x$, ... $x = \frac{6}{7}$.

II. — EQUATIONS IN WHICH THE UNKNOWN QUANTITY IS UNDER A RADICAL SIGN.

246. Rule. To clear an equation of a radical quantity, transpose this quantity to one side of the equation, and the rational terms to the other side; then involve both sides to a power corresponding to the radical sign.

If more than one term be under the radical sign, the preceding operation must be repeated.

EXAMPLES.

1. Let $\sqrt{(x-1)} - 1 = 2$, to find the value of x.

Transposing, $\sqrt{(x-1)} = 2 + 1 = 3$.

Squaring, $x - 1 = 9$;

and $x = 9 + 1 = 10$.

2. Given $\sqrt{(x-5)} - 3 = 4 - \sqrt{(x-12)}$, to find the value of x.

Transposing, $\sqrt{(x-5)} = 7 - \sqrt{(x-12)}$.

Squaring, $x - 5 = 49 - 14\sqrt{(x-12)} + x - 12$.

Transposing, $14\sqrt{(x-12)} = 49 - 12 + 5 = 42$.

Dividing by 14, $\sqrt{(x-12)} = 3$.

Squaring, $x - 12 = 9$

$\therefore x = 21$.

3. Let $\sqrt{\{5 + \sqrt{(x-4)}\}} - 2 = 1$, to find the value of x.

Transposing, $\sqrt{\{5 + \sqrt{(x-4)}\}} = 1 + 2 = 3$.

Squaring, $5 + \sqrt{(x-4)} = 9$.

Transposing, $\sqrt{(x-4)} = 9 - 5 = 4$.

Squaring, $x - 4 = 16$

$\therefore x = 16 + 4 = 20$.

4. Given $a + x = \sqrt{\{a^2 + x\sqrt{(c^2 + x^2)}\}}$, to find the value of x

Squaring, $a^2 + 2ax + x^2 = a^2 + x\sqrt{(c^2 + x^2)}$,

or $2ax + x^2 = x\sqrt{(c^2 + x^2)}$.

Dividing by x, $2a + x = \sqrt{(c^2 + x^2)}$.

Squaring, $4a^2 + 4ax + x^2 = c^2 + x^2$.

Taking x^2 from both sides, and transposing,

$$4ax = c^2 - 4a^2$$

$$\therefore x = \frac{c^2 - 4a^2}{4a}.$$

EXERCISES.

1. ... $\sqrt{(x+3)} = 4$, ... $x = 13$.

2. ... $\sqrt{(x-5)} = \sqrt{(x+2)} - 1$, ... $x = 14$.

3. ... $\sqrt{(12+x)} - 2 = \sqrt{x}$, ... $x = 4$.

4. ... $\sqrt{x(a+x)} = a - x$, ... $x = \dfrac{a}{3}$.

5. ... $\sqrt{\{6 + \sqrt{(x-1)}\}} = 3$, ... $x = 10$.

6. ... $\sqrt{x} - 2 = \sqrt{(x-8)}$, ... $x = 9$.

7. ... $a + x + \sqrt{a^2 + x^2} = b$, . . $x = \dfrac{b}{2}\left\{1 - \dfrac{a}{b-a}\right\}$.

8. ... $1 + \sqrt{1+x} = \sqrt{1 + x + \sqrt{1-x}}$, . $x = -\dfrac{24}{25}$.

9. ... $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = \sqrt{m}$, . . $x = \dfrac{2a\sqrt{m}}{1+m}$.

10. ... $m(ax-b)^{\frac{1}{3}} = n(cx + dx - f)^{\frac{1}{3}}$, . $x = \dfrac{bm^3 - fn^3}{am^3 - (c+d)n^3}$.

11. ... $\dfrac{1+x^3}{(1+x)^2} + \dfrac{1-x^3}{(1-x)^2} = a$, . . . $x = \left(\dfrac{a-2}{a+4}\right)^{\frac{1}{2}}$.

12. ... $\dfrac{(4x+1)^{\frac{1}{2}} + 2x^{\frac{1}{2}}}{(4x+1)^{\frac{1}{2}} - 2x^{\frac{1}{2}}} = 9$, ... $x = \dfrac{4}{9}$.

QUESTIONS PRODUCING SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

247. The first consideration in the solution of a question in equations is, how to form the equation, by the solution of which the unknown quantity is to be found. No certain rule can be given generally applicable, and at the same time sufficiently minute, so that the student must frequently depend on his own skill. In some questions, the enunciation of the conditions furnishes easily and immediately the required equation; but in other cases it is not the conditions themselves, but other conditions deduced from these, that are employed in forming the equation. In the former case, the conditions are said to be *explicit*, in the latter *implicit*. As a general rule, the following may be given:—

248. RULE. Denote the unknown quantity by some letter; epresent by algebraical signs the operations to be performed

upon it and the known quantities, agreeably to the conditions of the question, as if the object in view were to verify some particular value of the unknown quantity; and an equation will be thus formed, the solution of which will give the required value of the unknown quantity.

The changing of a question into an algebraical equation will, however, be much facilitated by attending to the following principles :—

1. If the sum of two numbers be given, and one of them be represented by x, then the other will be their sum *minus* x.

2. If the difference of two numbers be given, and the less be represented by x, the greater will be x *plus* the difference; and if the greater be represented by x, the less will be x *minus* the difference.

3. If the product of two numbers be given, and one of them be represented by x, the other will be the product divided by x.

4. If the quotient of two numbers be given, and one of them be represented by x, the other will be the quotient multiplied by x.

5. If an nth part of any quantity be taken away, $\frac{(n-1)}{n}$ parts will be left; thus, if from x one-fifth of itself be taken, $\frac{4}{5}x$ will be left; but if an nth part be added, the sum will be $\frac{n+1}{n}$ parts of the quantity, which results may therefore be put down at once.

EXAMPLES.

1. What number is that which, being added to its fourth part, the sum is $= 10$?

Let $x =$ the number required;

then the fourth part of x is expressed by $\frac{x}{4}$, and this added to x must give 10 for the sum, according to the condition in the question. The relation, therefore, between the unknown and the known quantities or numbers, are expressed by this equation,

$$x + \frac{x}{4} = 10.$$

Multiplying by 4, $\qquad 4x + x = 40.$

Collecting, $\qquad 5x = 40.$

Dividing by 5, $\qquad x = 8.$

The number is therefore 8. In order to verify this value, it

must be tried whether or not it satisfies the condition of the question; thus, putting 8 for x in the above, it becomes

$$8 + \frac{8}{4} = 10,$$

or $$32 + 8 = 40,$$

or $$40 = 40, \text{ an identity.}$$

This result being an identity, shews that the value is correct. It is evident, by comparing the equation above with this method of verification of the number 8, that x in the one, and 8 in the other, are related in exactly the same manner with the known quantities or numbers given in the question; so that the equation is formed exactly in the same manner as if the object were to verify any particular number.

2. What number is that from which, if one-fifth be taken, the remainder will be 8?

Let $x =$ the number,

then, since one-fifth part is taken from it, the quantity left is $\frac{4}{5}x$, which by the question $= 8$, or

$$\frac{4}{5}x = 8.$$

Multiplying by 5, $$4x = 40,$$

$$\therefore x = 10.$$

It will be found on trial that the number 10 fulfils the condition of the question.

As the reader must now be familiar with the various steps of transposing, collecting, &c. in the solution of an equation, it will be unnecessary to name these in future, as the steps of solution will be evident by inspection.

3. What number is that which, being added to its third part, the sum is equal to its half added to 10?

Let $x =$ the number;

then the number with its third part is $= x + \frac{x}{3}$, and its half added to 10 is $= \frac{x}{2} + 10$; but by the condition of the question, these are equal, or

$$x + \frac{x}{3} = \frac{x}{2} + 10.$$

Multiply by 6, $6x + 2x = 3x + 60$

$$8x - 3x = 60$$

$$5x = 60,$$

$$\therefore x = 12, \text{ the number required.}$$

4. Find two numbers such that their sum shall be $= 47$, and their difference $= 23$.

Let $x =$ the less number;

then $x + 23 =$ the greater, by the second condition of the question; and therefore, by the first condition,

$$x + x + 23 = 47,$$

or $$2x = 47 - 23 = 24,$$

$$\therefore x = 12,$$

and $$x + 23 = 12 + 23 = 35.$$

The two numbers, therefore, are 12 and 35.

5. Divide a line 12 feet long into three parts, such that the middle one shall be double the least, and the greatest triple the least.

Let $x =$ length in feet of the least part,

then $2x =$ ... ... middle ... ,

and $3x =$ ... ... greatest ... ;

and since these three parts must together be $= 12$,

then $$x + 2x + 3x = 12$$

$$6x = 12,$$

$$\therefore x = 2 \text{ the least};$$

hence $$2x = 4 \text{ ... middle},$$

and $$3x = 6 \text{ ... greatest}.$$

6. It is required to divide £470 among three persons, so that the second may have £10 more than the first, and the third £30 more than the second.

Let x = sum to be received by the first,

then $x + 10 =$ ... ... second,

and $x + 40 =$ ... ... third;

therefore, by the condition of the question,

$$x + (x + 10) + (x + 40) = 470,$$

or $$3x + 50 = 470$$

$$3x = 470 - 50 = 420,$$

$$\therefore x = 140 = \text{sum received by first,}$$

$$x + 10 = 150 = \text{... ... second,}$$

$$x + 40 = 180 = \text{... ... third.}$$

Proof, $$470 = \text{... ... all three.}$$

7. The sum of two numbers is 20, and the less is to the greater as 2 to three: required the numbers.

Let x = less number,

then $20 - x$ = greater,

and therefore $x : 20 - x = 2 : 3$;

hence (244) $$3x = 40 - 2x$$

$$5x = 40,$$

$$\therefore x = 8 = \text{the less number,}$$

and $$20 - x = 20 - 8 = 12 = \text{the greater.}$$

This example contains an *implicit* condition—namely, the *proportion*, from which the equation is deduced.

8. The difference between two numbers is 2, and their product exceeds the square of the less by 8: what are the numbers?

Let x = the less,

then $x + 2 =$ ... greater;

and by the question,

$$x(x + 2) = x^2 + 8,$$

or $$x^2 + 2x = x^2 + 8.$$

Taking x^2 from both sides, $2x = 8$,

$$\therefore x = 4 = \text{the less,}$$

and $$x + 2 = 6 = \text{... greater.}$$

These values verify the equation, for

$$4(4+2) = 4^2 + 8,$$

or
$$16 + 8 = 16 + 8,$$

which is an identity.

9. A cistern was found to be one-third full of water, and after running into it 21 gallons, it was then found to be half full: required its capacity in gallons.

Let $x =$ the number of gallons the cistern can contain; then, by the question,

$$\frac{x}{3} + 21 = \frac{x}{2},$$

or
$$2x + 126 = 3x,$$

$$\therefore x = 126 = \text{its content in gallons.}$$

10. A cistern is supplied with water by two pipes; the less alone can fill it in 40 minutes, and the greater in 30 minutes: in what time will they both fill it when running together?

Let $x =$ the number of minutes in which both can fill it; then, since the first can fill the whole in 40 minutes, it can fill $\frac{1}{40}$ in a minute, and therefore $\frac{x}{40}$ in x minutes; in the same manner it can be shewn, that $\frac{x}{30}$ is the part filled by the second pipe in x minutes. But the sum of these two parts is the whole content of the cistern which $= 1$. Hence

$$\frac{x}{40} + \frac{x}{30} = 1.$$

Multiplying by 120,
$$3x + 4x = 120$$
$$7x = 120$$
$$x = 17\frac{1}{7} \text{ minutes,}$$

the time taken by both to fill the cistern.

11. One labourer (A) can perform a piece of work in 5 days, and another (B) can do the same in 6 days, and another (C) in 8 days: in what time can they perform the same work when the three are engaged in it together?

Let $x =$ number of days in which the three together can perform the work, then since A can do the whole in 5 days, he can do $\frac{1}{5}$ in one day, and therefore $\frac{x}{5}$ in x days; in the same manner it may be shewn, that B can do $\frac{x}{6}$ in x days, and C can do $\frac{x}{8}$ in x days; therefore A, B, and C can together do $\frac{x}{5}+\frac{x}{6}+\frac{x}{8}$ in x days, but by the question this is equal to the whole work $= 1$;

therefore $$\frac{x}{5}+\frac{x}{6}+\frac{x}{8}=1.$$

Multiplying by 120, $$24x+20x+15x=120,$$
$$59x=120,$$
$$\therefore x=2\frac{2}{59} \text{ days.}$$

12. How many pounds of tea at 5 shillings and 9 shillings per pound must be mixed to make a box of 200 lbs. at 6 shillings a pound?

Let $x =$ number of pounds at 5 shillings,

then $200-x =$ ... ... 6 ... ;

also $5x =$ the value of the former,

and $9(200-x) =$ ... ... latter,

and $1200 =$ ... ... mixture.

But the value of the two kinds of tea must just be equal to the value of the mixture;

therefore $$5x+9(200-x)=1200,$$
$$5x+1800-9x=1200,$$
or $$4x=600,$$
$$\therefore x=150= \text{number of lbs. at 5s.}$$
and $$200-x=50= \text{ ... ... at 9s.}$$

13. A person at play lost one-fourth of his money, and then gained 5 shillings; he then lost the half of what he now had, and found that he had only 7 shillings remaining: with what sum did he begin to gamble?

Let $x =$ number of shillings he at first had; then, after losing

one-fourth of his money, and gaining 5 shillings, he had just

$$x - \frac{x}{4} + 5 = \frac{3x}{4} + 5;$$

since he now lost the half of this sum, he had remaining the other half; therefore by the question,

$$\frac{1}{2}\left(\frac{3x}{4} + 5\right) = 7,$$

or $$\frac{3x}{4} + 5 = 14,$$

or $$\frac{3x}{4} = 9;$$

hence $$3x = 36,$$

$$\therefore x = 12.$$

The equation may also be stated thus:—

$$x - \frac{x}{4} + 5 - \frac{1}{2}(x - \frac{x}{4} + 5) = 7,$$

or $$\frac{3x}{4} - \frac{1}{2}\left(\frac{3x}{4} + 5\right) = 7 - 5 = 2.$$

Multiplying by 8, $$6x - 3x - 20 = 16,$$

$$3x = 36,$$

$$\therefore x = 12.$$

14. The sum of two numbers is $= s$, and their difference $= d$: what are the numbers?

Let $x =$ the greater,

then $x - d =$ the less,

and $$x + (x - d) = s,$$

or $$2x - d = s,$$

or $$2x = s + d,$$

$$\therefore x = \frac{1}{2}(s + d) = \frac{s}{2} + \frac{d}{2},$$

and $$x - d = \frac{s}{2} + \frac{d}{2} - d = \frac{s}{2} - \frac{d}{2}.$$

From this result is derived the following useful theorem:—

249. Theorem. Half the difference of two quantities added to half their sum = the greater, and taken from half the sum = the less.

15. The hour and minute hands of a watch are exactly together between 8 and 9 o'clock: required the exact time at which they coincide.

Let the number of minutes more than 40 be denoted by x, or $x =$ minutes from VIII to the point of coincidence m. Then the hour-hand moves from VIII to the point of coincidence m, during the time that the minute-hand moves from XII to the same point, or the former hand moves over x minutes, while the latter moves over $40 + x$ minutes; but the latter hand moves 12 times faster than the former; therefore

$$40 + x = 12x,$$

or

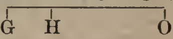
$$11x = 40,$$

$$\therefore x = \frac{40}{11} \text{ minutes} = 3^{m}\ 38^{s}\ \frac{2}{11}.$$

Hence the required time is $43^{m}\ 38^{s}\ \frac{2}{11}$ past 8 o'clock.

16. A hare is 40 of her own leaps before a greyhound, and takes 5 leaps for the greyhound's 4; but 3 of the greyhound's leaps are equal to 4 of the hare's: how many leaps must the greyhound take to catch the hare?

Let $x =$ the number of leaps the hare takes before being caught, or from H to O, H being the hare's starting-point; then $x + 40 =$ the number of leaps of the hare from the greyhound's starting-point (G) to the place where he overtakes the hare (O).



But $5 : 4 = x : \frac{4x}{5}$, the *number* of leaps taken by the greyhound after starting till he overtakes the hare, for the hare takes the same time to run over HO that the greyhound takes to run over GO, and their number of leaps in the same time is as 5 to 4.

Further, the whole number of leaps of the hare and greyhound in the same space, as GO, is as 4 to 3, or inversely as their length, and the number of their leaps in the space GO are respectively $x + 40$ and $\frac{4x}{5}$; therefore

$$x + 40 : \frac{4x}{5} = 4 : 3,$$

$$3x + 120 = \frac{16x}{5},$$

or

$$15x + 600 = 16x,$$

therefore

$$x = 600 = \text{number of hare's leaps in HO},$$

and

$$\frac{4x}{5} = \frac{4 \times 600}{5} = 480 = \text{number of greyhound's leaps in GO}.$$

17. A cistern is supplied with water by one pipe, and emptied by another; it can be filled by the former in 20 minutes, and emptied by the latter in 15; supposing the cistern at first to be full, in what time would it be emptied when both pipes are running?

Let $x =$ the number of minutes,

then $20 : x = 1 : \frac{x}{20}$, the portion filled in x minutes,

and $15 : x = 1 : \frac{x}{15}$, ... run out ... ;

also, since the excess of the latter portion above the former is = the content of the cistern,

$$\frac{x}{15} - \frac{x}{20} = 1$$

$$20x - 15x = 300$$

$$5x = 300,$$

$$\therefore x = 60 \text{ minutes} = 1 \text{ hour}.$$

18. A person bought a number of sheep at 16 shillings a head, but found that he had not enough of money to pay for them, by 40 shillings; had he, however, given only 15 shillings for each, he would have had 60 shillings over: how many sheep did he purchase, and how much money had he?

Let $x =$ the number of sheep,

then by the question, $16x - 40 = 15x + 60$,

or $x = 100 =$ number of sheep,

and $1600 - 40 = 1560\text{sh.} = £78 =$ his money.

19. A bill of £700 was paid in sovereigns, half sovereigns, and crowns, and an equal number of each was used: required this number.

Let $x =$ the number of each,

then $20x + 10x + 5x = 700 \times 20 = 14000$ shillings,

or $35x = 14000$,

$\therefore x = 400 =$ the number of each.

20. A labourer was engaged for 30 days at 15 pence a day with maintenance; but for every day he was idle there was to be deducted 10 pence for maintenance. After his engagement expired, the sum he received was 25 shillings: how many days did he work, and how many was he idle?

Let $x =$ number of days he worked,

then $30 - x = \quad \ldots \quad \ldots$ he was idle;

also $15x =$ wages due for his work,

and $10(30 - x) =$ sum to be deducted;

therefore $15x - 10(30 - x) = 25 \times 12 = 300$ pence,

or $15x - 300 + 10x = 300$,

$$25x = 600,$$

$$\therefore x = 24 = \text{the days he worked,}$$

and $30 - x = 6 = \quad \ldots$ he was idle.

250. Questions in which the given quantities are numbers, may be generalised, by assuming literal quantities for the known ones, and then the solution of the question will afford a *rule* for the solution of all similar ones.

The following question is the third of the preceding, generalised:—

21. Required a number such that, being added to its mth part, the sum shall be equal to its nth part added to a number a.

Let $x =$ the number,

then $\frac{x}{m}$ is the expression for its mth part,

and $\frac{x}{n} \quad \ldots \quad \ldots$ nth part;

therefore $x + \frac{x}{m} = \frac{x}{n} + a$;

multiplying by mn

$$mnx + nx = mx + amn$$

$$mnx + nx - mx = amn$$

$$(mn - m + n)x = amn$$

and hence

$$x = \frac{amn}{mn - m + n}.$$

This value of x serves as a rule for the solution of all similar problems in which particular numerical values are given for a, m, and n. Thus, for the 3d example,

$$a = 10, m = 3, n = 2;$$

therefore $x = \dfrac{10 \times 3 \times 2}{3 \times 2 - 3 + 2} = \dfrac{60}{5} = 12.$

In a similar manner the 11th example may be generalised thus:—

A can perform a piece of work w in a days, B can accom-

plish the same in b days, and C in c days: in how many days wil they finish the work when all are engaged?

Let $x =$ the number of days, and $w =$ the whole work; then sinc A could perform the work w in a days, he could do a $\frac{w}{a}$ part i one day, and therefore A could do $\frac{wx}{a}$ in x days; in the sam manner it can be shewn that B could do $\frac{wx}{b}$ in x days, and that C could do $\frac{wx}{c}$ in x days: but since the sum of these three parts = the whole work, therefore

$$\frac{wx}{a} + \frac{wx}{b} + \frac{wx}{c} = w,$$

or $$\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 1;$$

multiplying by abc, $$bcx + acx + abx = abc$$

$$(ab + ac + bc)x = abc,$$

$$\therefore = \frac{abc}{ab + ac + bc}.$$

In the 11th example, $a = 5$, $b = 6$, $c = 8$,

therefore $x = \frac{5 \times 6 \times 8}{30 + 40 + 48} = \frac{240}{118} = \frac{120}{59} = 2\frac{2}{59}$.

EXERCISES.

1. What number added to its fifth part $= 24$, . . $= 20$

2. Required a number which exceeds its fourth part by 27, $= 36$

3. What is that number, which being added to $7 = 5$ times it fourth part? . . . . $= 28$

4. Required the number which exceeds its sixth part as muc as 26 exceeds its fourth part? . . . . . . $= 24$

5. The sum of two numbers is $= 20$, and their difference is $= 8$ what are these numbers? . . . . . . $= 14$ and 6

6. At an election, the number of votes given for two candidate was $= 256$; the successful candidate had a majority of 50 vote how many voted for each of the candidates? . $= 153$ and 103

7. Divide a line of 24 inches into two unequal parts, so that the greater may exceed the less by 10 inches, = 17 and 7.

8. Divide a line of 42 feet in length into two parts, so that the one shall be double the other, . . . = 14 and 28.

9. Divide the number 30 into 3 such parts, that the second may be double of the first, and the third triple the same, = 5, 10, and 15.

10. Divide a line of 60 inches into 3 such parts, that the second may be = 3 times the first, and the third double the second, = 6, 18, and 36.

11. Divide the number 252 into three parts, such that one-third of the first, one-fourth of the second, and one-fifth of the third, shall all be equal to one another, . . = 63, 84, and 105.

12. An uncle divided £375 among three nephews. To the first he gave four-fifths of the sum given to the second, and the third received a fifth part of the second's share more than the second: how much did each receive? . . . = 100, 125, and 150.

13. The sum of two numbers is = 60; and the less is to the greater as 5 to 7: what are these numbers? . = 25 and 35.

14. The sum of £154 was paid in half sovereigns, half crowns, and groats (4 pence each), and an equal number of each of these coins was used: what was the number? . . . = 240.

15. Two canal-boats are despatched from the same place, the first at 6 o'clock in the morning, the other at 4 in the afternoon; the first runs 4 miles an hour, and the second 9: in how many hours will the second boat overtake the first? . = 8 hours.

16. The sum of two numbers is 10, and their product is equal to the excess of 60 above the square of the greater: required the numbers, ... = 4 and 6.

17. Divide the number 20 into two parts, so that 5 times the greater may exceed 6 times the less by 12, . . = 12 and 8.

18. A cistern was found to be three-fourths full of water, but after running off 220 gallons, it was found to be one-fifth full: how many gallons could it contain? . . . . = 400.

19. A cistern supplied by 3 pipes can be filled by the first in 10 minutes, by the second in 12, and by the third in 15 minutes: in what time would it be filled by the three running simultaneously? = 4 minutes.

20. A cistern can be filled by one pipe in 16 minutes, and emptied by another in 20 minutes: supposing it at first empty, what time would it be filled when both pipes are running? = 80 minutes.

21. Divide 48 into four such parts that the first *plus* 3, the second *minus* 3, the third multiplied by 3, and the fourth divided by 3, may be all equal to each other, . . = 6, 12, 3, and 27.

22. A person has a certain number of shillings in each hand, and if he take 8 from the left, and put them in the right hand, he would then have 4 times as many in his right hand as in his left; but at first he had 5 more in his right hand than in his left: how many had he in each at first? . . . . = 15 and 20.

23. The hour and minute hands of a watch are observed to coincide between 4 and 5 o'clock: how many minutes is it past 4?

= 21 minutes 49 $\frac{1}{11}$ seconds.

24. A labourer engages to work for 3s. 6d. a day with board, but to allow 9d. for his board each day that he is unemployed. At the end of 24 days he has to receive £3, 2s. 9d.: how many days has he wrought? . . . . = 19.

25. Three workmen are employed to dig a ditch of 191 yards in length. A can dig 27 yards in 4 days, B 35 yards in 6 days, and C 40 yards in 12 days: in what time could they do it if they worked simultaneously? . . . . . = 12 days.

26. Two persons depart at the same time from London and Edinburgh, and travel till they meet; the one goes 20 miles a day and the other 30: in how many days will they meet, the distance being 400 miles? . . . . = 8 days.

27. Two persons (A and B) depart from the same place to go in the same direction; B travels at the rate of 2 miles an hour, and A 3 miles, but B has the start of A by 5 hours: in how many hours will A overtake B? . . . . . = 10 hours.

28. Two persons (A and B) depart at the same time from the same place, to travel in the same direction round an island 36 miles in circumference; A travels 3 miles an hour, and B $2\frac{1}{2}$: after how many hours shall they come together? = 72 hours.

29. Divide £4400 among three persons, so that the first may have three-fifths of the second's share, and the second three-fourths of the third's share, . . . = £900, £1500, and £2000.

30. A hare has a start of 80 of its own leaps before a greyhound, and takes 3 leaps for every 2 taken by the greyhound, but one of the greyhound's leaps is equal to 2 of the hare's: how many leaps will the hare have taken before it is caught? . . = 240.

31. A and B engage in trade on the same capital; A gains 160 pounds, and B loses 190, but A's money is now 8 times B's: with how much money did they begin? . . . = £240.

32. Divide a number a into two parts, so that b times the greater shall exceed c times the less by d,

$$\text{The less} = \frac{ab - d}{b + c}, \text{ the greater} = \frac{ac + d}{b + c}.$$

33. A person has a hours at his disposal: how far may he travel in a coach which goes b miles an hour, so as to return home in time, walking back at the rate of c miles an hour? Also find the number of miles if $a = 2$, $b = 12$, $c = 4$,

$$1. = \frac{abc}{b + c} \text{ miles.} \quad 2. = 6 \text{ miles.}$$

NEGATIVE SOLUTION OF SIMPLE EQUATIONS.

251. The value of an unknown quantity in an equation is sometimes found to be negative; it will be necessary, therefore, to consider the meaning of such a solution. A negative result would at first appear to be absurd, and, in its literal sense, it is so; but a proper interpretation may be easily determined, by means of the conventional use of the negative sign (44). To illustrate this negative result, the following example may be given:—

252. Let it be required to find a number such that if a given number c be added to it, the sum shall be a given number a.

$$\text{If } x = \text{the required number,}$$

$$\text{then } x + c = a$$

$$\text{or } \quad x = a - c.$$

So long as a exceeds c, the value of x is positive; but let a be less than c, and x becomes negative; thus,

$$\text{let } a = 10,\ c = 8, \text{ then } x = 10 - 8 = 2; \text{ but,}$$

$$\text{let } a = 5,\ c = 9, \text{ then } x = 5 - 9 = -4.$$

In the second case, x has a negative value, which implies that, instead of adding c to x, x must be taken from c and added to a; and hence the question, in its literal sense, is impossible with these values of a and c, or generally, when $c > a$. This sign, therefore, apprises us of the necessity of modifying the condition of the question, in order to adapt it to the case of $a < c$. For such a case, the condition, to be literally fulfilled, must be expressed thus:—

Required a number such that if it be subtracted from a given number c, the remainder shall be = another given number a.

This question will be solved for the case of $a = 5$, $c = 9$, if $x = 4$, for then $9 - 4 = 5$; and the general equation will be

$$c - x = a$$

$$x = c - a = 9 - 5 = 4.$$

Instead, however, of making two distinct questions for the two cases of $c \angle a$ and $c > a$, one question and one solution will serve for both, by considering that the subtracting of a positive quantity is equivalent to adding a negative one (56), or that, in this case, the subtracting of $x = 4$ in this question is the same as adding $x = -4$ in the preceding one, when $c > a$; therefore, with this understanding, the solution of the former question

$$x = c - a$$

comprehends also the solution of the latter. Thus, when

$$a = 5,\ c = 9, \text{ and } x = a - c = -4,$$

if c be added to x, by the ordinary rule of algebraic addition, the result is

$$9 + [-4] = 9 - 4$$

$$c + x = 9 - 4 = 5 = a.$$

253. In order to explain more fully the meaning of negative solutions, let the equation

$$ax + d = cx + b$$

be given. From this equation,

$$x = \frac{b - d}{a - c}.$$

Several cases requiring some consideration will occur according to the relative values of the given quantities a, b, c, d.

I. Let $b > d$ and $a > c$, then x is positive.

II. If, while one of the quantities $(b - d)$ or $(a - c)$ is positive, the other be negative, then the value of x is negative, and the equation becomes

$$-ax + d = -cx + b.$$

1. When $b \angle d$ and $a > c$, $(b - d)$ is negative, and $(a - c)$ positive, so that

$$x = \frac{d - b}{a - c}.$$

In order that the value of x may be positive, the terms in the given equation containing x are therefore in this case to be made

negative, which is the same as subtracting ax and cx; but subtracting these is simply adding, by the rule of addition, $-ax$ and $-cx$, or a and c multiplied by $-x$; that is, by $-\frac{(d-b)}{a-c}$, or $\frac{-(d-b)}{a-c}$, or $\frac{b-d}{a-c}$, which is negative, because $b-d$ is so, since $b \angle d$. But this is exactly the value of x in the first case; and hence the solution in that case comprehends that in this, observing the conventional definition (42), and the principle in (56).

2. When $b > d$ and $a \angle c$, the result will be found to be the same as the preceding, and may be obtained in the same manner, by taking $a-c$ for $-(c-a)$.

III. When $b = d$, and, at the same time, c not $= a$, then

$$x = \frac{0}{a-c} = 0,$$

as is evident from the original equation, which becomes

$$ax - cx = 0,$$

or

$$(a-c)x = 0.$$

Dividing by $(a-c)$,

$$x = 0.$$

IV. When b is not $= d$, but $a = c$, then

$$x = \frac{b-d}{a-c} = \frac{b-d}{0} = \infty;$$

the equation in this case being

$$(a-c)x = b-d,$$

or

$$0 \times x = b-d;$$

and since no finite value of x, multiplied by 0, can give a finite product $= b-d$, x must $= \infty$.

V. When $a = c$, and $b = d$,

$$x = \frac{0}{0},$$

from which no particular value of x can be found, or its value is indeterminate. The equation in this case is

$$(a-c)x = b-d,$$

or

$$0 \times x = 0;$$

and since any number, multiplied by 0, gives 0, therefore any

value whatever of x fulfils the equation, or it has no particular value.

254. The meaning, therefore, of the expression $\frac{0}{0}$ is merely the indeterminateness of the quantity whose value it expresses; it does not convey any sign of absurdity or impossibility in the equation.

255. An equation whose members are the same quantities expressed either in the same or in different forms, is called an *identical* equation.

The last case of the preceding equation, namely,

$$ax + b = cx + d,$$

or

$$ax + b = ax + b,$$

since $a = c$ and $b = d$, is an equation of this kind; and so are the equations

$$x(x-2) = x^2 - 2x$$

$$(x-a)(x-b) = x^2 - (a+b)x + ab$$

$$3x(2-4) + 36 = 6x - 12x + 36.$$

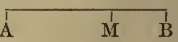
It is evident that whatever value is assigned to the unknown quantity in an identical equation, its two members are equal; so that the equation is satisfied by any value of this quantity, and therefore no particular value of it can be assigned.

256. The nature of a negative solution is very clearly illustrated by the following question:—

Two couriers depart at the same time from two towns (A and B), distant by a miles from each other; the former travels m miles an hour, and the latter n miles: where shall they meet?

There are two cases of this question:—

I. When the couriers go in *opposite directions.*



Let M be the point where they meet, and $a = AB$ the distance between the places,

$x = AM$, the distance that A travels,

then $a - x = MB$ ... ... B ... ;

also $\frac{x}{m} =$ the number of hours that A travels,

and $\frac{a-x}{n} =$ ... ... B ... ;

but A and B travel the same number of hours, therefore

$$\frac{x}{m} = \frac{a - x}{n}$$

$$nx = am - mx$$

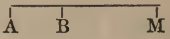
$$x = \frac{am}{m + n},$$

and $$a - x = \frac{an}{m + n}.$$

Whatever values be given to a, m, n, that of x will in this case be always positive, and no difficulty occur.

II. When the couriers go in the *same direction.*

As in the former case, let M be the point of meeting, A and B travelling in that direction, and let



$$a = \text{AB},$$

and $$x = \text{AM},$$

whence $$x - a = \text{BM};$$

then, as in the preceding case,

$$\frac{x}{m} = \frac{x - a}{n}$$

$$nx = mx - am$$

$$x = \frac{am}{m - n},$$

and $$x - a = \frac{an}{m - n}.$$

It appears from this value of x, that, so long as $m > n$, or A's rate of travelling exceeds B's, the value of x will be positive, and there will be no difficulty in interpreting its value.

But suppose that $m < n$, the value of x is negative, and the equation

$$nx = mx - am$$

now becomes

$$-nx = -mx - am$$

$$x = \frac{am}{n - m}.$$

This negative value implies an impossibility in the question

understood literally, and indicates the necessity of modifying its enunciation; but, by the following considerations, this will be found unnecessary :—

To subtract $\frac{am}{n - m}$ is the same as to add $\frac{am}{-(a - m)}$, by the rules of algebraic addition (56); and $\frac{am}{-(n - m)} = \frac{am}{m - n}$. But this quantity being in the present case *negative*, as $m \angle n$, the value of x is also negative, and being the numerical value of a *line*, its value must be measured in a direction *opposite* to that of AM (44), namely, in the direction AM′; hence AM′

$$= x = \frac{am}{m - n},$$

and M′ is the point where A and B would meet.

Were this negative value of x to be measured from B, it would be identical with that of x in the first case, m and n being interchanged. For in this case the value of x will be measured from B, provided AB be added to it. But x being in this case negative, AB must be taken as negative, and $= -a$; then

$$x = \frac{am}{m - n} - a = \frac{am - am + an}{m - n} = \frac{an}{m - n}, \text{ or } x = -\frac{an}{n - m},$$

which is exactly equivalent to the interchanging of m and n in the first value of x, in this case, and taking x negative.

When the given quantities in a question are general, that is, when they are expressed by letters, interesting consequences may sometimes be deduced from the solution, by assigning particular values to the letters.

Thus, from the preceding value of x in the first case,

$$x = \frac{am}{m + n},$$

when $$m = n,\ x = \frac{am}{m + m} = \frac{am}{2m} = \frac{a}{2};$$

that is, each of the couriers travels over precisely half the distance, as is otherwise evident.

When $n = 0$, $x = \frac{am}{m} = a$,

or A has the whole distance to travel $=$ AB.

When $m = 0$, $x = \frac{a \times 0}{0 + n} = \frac{0}{n} = 0$,

or A remains stationary; B meets him at A, or B travels the whole distance $=$ BA, as is otherwise evident.

In the second case, when $m = n$, $x = \frac{am}{m - n} = \frac{am}{0} = \infty$;

that is, A must travel an indefinitely great distance, or, in other words, he never can overtake B, as is manifest from the equality of their rates of travelling.

When $n = 0$, $x = \frac{am}{m} = a$, and A travels to B.

When $n = \frac{m}{2}$, $x = \frac{2am}{m} = 2a$, or A travels twice the length of AB.

When $n = 2m$, $x = \frac{am}{-m} = -a$, or A travels in an opposite direction from A towards M′, a distance $= a$, and consequently B a distance $= 2a$.

II.—SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

257. Sometimes a question requires for its solution the determination of the values of two unknown quantities. In this case two conditions are stated. These conditions are *necessary*, and they are *sufficient;* they must also be wholly *independent* of each other. They must likewise be *consistent*, for if contradictory, the solution would lead to an absurd conclusion. And, lastly, they may be either *implicit* or *explicit*.

Questions of this kind may, however, sometimes be solved by using only one unknown quantity, as in some examples in the former section containing two conditions, in which one condition is used in determining the notation, and the other in obtaining an equation. (See examples 4 and 7, of art. 248.)

258. The first part of the process of solution is to *eliminate* one of the unknown quantities; that is, by some combination of the two equations to derive a new one excluding one of the unknown quantities: this resulting equation, containing only the other unknown quantity, may be solved by the former method for such equations.

The following is an example of dependent equations:—

$$3y = 2x - 4$$

$$6y = 4x - 8,$$

the latter being derived from the former, merely by multiplying its terms by 2.

The two equations, $2x - y = 5,$

and $2x - y = 6,$

are inconsistent equations. It is evident that the same values of x and y that satisfy the former cannot also fulfil the latter; they are therefore contradictory.

259. There are three methods of solving two equations containing two unknown quantities; namely, *equating*, *substitution*, and *equalising coefficients.*

I. — BY EQUATING TO SOLVE TWO EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

260. Rule. Find a value of one of the unknown quantities in each of the equations, and equate these values; that is, make them the members of a new equation, which will then contain only one unknown quantity, the value of which may be found as before.

The simplest method in this process is to find a value of the unknown quantity which is least involved by coefficients in the given equation.

EXAMPLES.

1. If $\left\{\begin{array}{l} x + y = 8 \\ x - y = 2 \end{array}\right\}$ what are the values of x and y?

By finding a value of y in each equation, we have

in the first equation, ... $y = 8 - x$ ... [3]

in the second equation, ... $y = x - 2$ ... [4]

Equating these values of y,

$$x - 2 = 8 - x,$$

or $$2x = 10,$$

$$\therefore x = 5.$$

By finding similarly a value of x in each of the given equations, and equating these values, which will be expressed in terms of y, the value of y may be found in the same way as that of x above. But if the value of x already found, namely 5, be substi-

tuted in either of the given equations, or in [3] or [4], the equation will then contain only y, whose value may be found. Thus, substituting 5 for x in [3], it becomes

$$y = 8 - x = 8 - 5 = 3;$$

or, substituting this value of x in [4], it becomes

$$y = x - 2 = 5 - 2 = 3,$$

which gives the same value of y.

2. If $\left\{\begin{array}{l} 2x - 3y = -1 \\ 3x - 2y = 6 \end{array}\right\}$ what are the values of x and y?

In the first equation, $y = \dfrac{1 + 2x}{3}$ ... [3]

In the second, $y = \dfrac{3x - 6}{2}.$

Equating, $\dfrac{1 + 2x}{3} = \dfrac{3x - 6}{2}$

$$2 + 4x = 9x - 18$$

$$5x = 20,$$

$$\therefore x = 4,$$

and by [3], $y = \dfrac{1 + 8}{3} = \dfrac{9}{3} = 3.$

EXERCISES.

1. If $\left\{\begin{array}{l} x - y = 2 \\ x + y = 6 \end{array}\right\}$ . . . . . . $\left\{\begin{array}{l} x = 4. \\ y = 2. \end{array}\right.$

2. ... $\left\{\begin{array}{l} 5x - 2y = 4 \\ 2x - y = 1 \end{array}\right\}$ . . . . . $\left\{\begin{array}{l} x = 2. \\ y = 3. \end{array}\right.$

3. ... $\left\{\begin{array}{l} \dfrac{x}{2} - \dfrac{y}{3} = 2 \\ \dfrac{x}{3} - y = -1 \end{array}\right\}$ . . . . . $\left\{\begin{array}{l} x = 6. \\ y = 3. \end{array}\right.$

... $\left\{\begin{array}{l} \dfrac{x}{4} - \dfrac{y}{4} = 1 \\ \dfrac{x}{3} + \dfrac{y}{2} = 8 \end{array}\right\}$ . . . . . $\left\{\begin{array}{l} x = 12. \\ y = 8. \end{array}\right.$

5. If $\left\{\begin{aligned} \frac{x}{9} - \frac{y}{8} &= 1 \\ \frac{x}{6} + \frac{y}{4} &= 12 \end{aligned}\right\}$ . . . . . $\left\{\begin{aligned} x &= 36. \\ y &= 24. \end{aligned}\right.$

6. ... $\left\{\begin{aligned} \frac{x}{5} + \frac{y}{2} &= 14 \\ \frac{x}{9} - \frac{y}{5} &= 3 \end{aligned}\right\}$ . . . . . $\left\{\begin{aligned} x &= 45. \\ y &= 10. \end{aligned}\right.$

II. — BY SUBSTITUTION TO SOLVE TWO EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

261. Rule. Find a value of one of the unknown quantities in either of the equations; substitute this value in the other equation; and a new equation will thus be formed, containing only one unknown quantity.

The solution will be more simple if the unknown quantity whose value is taken be that which is least involved in the given equations.

EXAMPLES.

1. If $\left\{\begin{aligned} y - x &= 1 \\ x + y &= 5 \end{aligned}\right\}$ what are the values of x and y?

In the first equation $\quad y = 1 + x \quad ... \quad [3].$

Substituting this value for y in the second equation,

$$x + 1 + x = 5$$

$$2x = 4,$$

$$\therefore x = 2;$$

and by [3], $\quad y = 1 + 2 = 3.$

2. If $\left\{\begin{aligned} \frac{x}{5} + \frac{y}{4} &= 2 \\ x - y &= 1 \end{aligned}\right\}$ what are the values of x and y?

Finding a value of y in the second equation, as it is least involved in it,

$$y = x - 1 \quad ... \quad [3],$$

substituting this value in the first $\frac{x}{5} + \frac{x-1}{4} = 2$,

$$4x + 5x - 5 = 40$$

$$9x = 45,$$

$$\therefore x = 5,$$

and by [3], $y = 5 - 1 = 4.$

EXERCISES.

1. If $\left\{\begin{array}{l} \frac{x}{3} - \frac{y}{6} = 1 \\ \frac{x}{4} - \frac{y}{9} = 1 \end{array}\right\}$ . . . . . $\left\{\begin{array}{l} x = 12. \\ y = 18. \end{array}\right.$

2. ... $\left\{\begin{array}{l} x - y = 10 \\ \frac{x}{5} - \frac{y}{3} = 0 \end{array}\right\}$ . . . . $\left\{\begin{array}{l} x = 25. \\ y = 15. \end{array}\right.$

3. ... $\left\{\begin{array}{l} 2x - 3y = -4 \\ x - \frac{y}{3} = 12 \end{array}\right\}$ . . . . $\left\{\begin{array}{l} x = 16. \\ y = 12. \end{array}\right.$

4. ... $\left\{\begin{array}{l} \frac{y}{5} - \frac{x}{4} = 1 \\ 5x - 3y = 10 \end{array}\right\}$ . . . . $\left\{\begin{array}{l} x = 20. \\ y = 30. \end{array}\right.$

The examples and exercises under the first case may be solved by the second as additional exercises.

III.—BY EQUALISING THE COEFFICIENTS TO SOLVE TWO EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

262. Rule I. Multiply both equations, if necessary, by such numbers as will make the coefficients of one of the unknown quantities the same in both equations; then by finding the difference or sum of the equations, according as the two equal terms have like or unlike signs, these two terms will destroy each other; and the resulting equation will contain only one unknown quantity.

The coefficients of an unknown quantity in the two equations may be equalised by multiplying each equation by the coefficient of the unknown quantity in the other. When these two coeffi-

cients, however, are not prime, find their least common multiple; and then, by multiplication, make the two coefficients equal to this multiple.

If the equations have fractional coefficients, the fractions ought to be cleared away before applying the rule.

EXAMPLES.

1. If $\left.\begin{cases} 3x + 2y = 21 \\ x - 2y = -1 \end{cases}\right\}$ what are the values of x and y?

The coefficients of y being equal in these equations, it will be unnecessary to multiply them; and as the signs are unlike, the equations must be added, which gives

$$4x = 21 - 1 = 20,$$

$$\therefore x = 5.$$

The value of y might be similarly found by equalising the coefficients of x; but it will be more readily found by substituting the value of x, as already found, in one of the given equations. Since the second is the less involved, it may be chosen for this purpose. The result is,

$$5 - 2y = -1$$

$$2y = 6,$$

$$\therefore y = 3.$$

2. If $\left.\begin{cases} 3x - 2y = 7 \\ 5y - 2x = 10 \end{cases}\right\}$ what are the values of x and y?

As x is least involved, its coefficients should be equalised, by multiplying the first equation by 2, and the second by 3,

$$6x - 4y = 14$$

$$15y - 6x = 30;$$

adding these, $$15y - 4y = 44$$

$$11y = 44,$$

$$\therefore y = 4;$$

and, by the first equation, $x = \dfrac{7 + 2y}{3} = \dfrac{7 + 8}{3} = \dfrac{15}{3} = 5.$

3. If $\left\{\begin{array}{l}\dfrac{x}{4}+\dfrac{y}{5}=8\\[2ex]\dfrac{x}{5}+\dfrac{y}{3}=9\end{array}\right\}$ what are the values of x and y?

Clearing the equations of fractional coefficients,

the first becomes $\qquad 5x+4y=160 \quad \ldots \quad [3],$

the second ... $\qquad 3x+5y=135 \quad \ldots \quad [4];$

multiplying [3] by 5, $\qquad 25x+20y=800,$

... [4] by 4, $\qquad 12x+20y=540,$

subtracting the latter from the former, $\qquad 13x=260,$

$$\therefore x=20;$$

and by [3], $y=40-\dfrac{5x}{4}=40-\dfrac{100}{4}=40-25=15.$

EXERCISES.

1. If $\left\{\begin{array}{l}2x-y=3\\3x+2y=22\end{array}\right\}$ . . . . $\left\{\begin{array}{l}x=4.\\y=5.\end{array}\right.$

2. ... $\left\{\begin{array}{l}3x+2y=19\\2x-3y=4\end{array}\right\}$ . . . . $\left\{\begin{array}{l}x=5.\\y=2.\end{array}\right.$

3. ... $\left\{\begin{array}{l}45x+8y=350\\21y-13x=132\end{array}\right\}$ . . . . $\left\{\begin{array}{l}x=6.\\y=10.\end{array}\right.$

4. ... $\left\{\begin{array}{l}\dfrac{x}{8}+\dfrac{y}{9}=42\\[2ex]\dfrac{x}{9}+\dfrac{y}{8}=43\end{array}\right\}$ . . . . $\left\{\begin{array}{l}x=144.\\y=216.\end{array}\right.$

5. ... $\left\{\begin{array}{l}\dfrac{x}{2}-\dfrac{y}{3}=3\\[2ex]\dfrac{x}{6}+\dfrac{y}{9}=3\end{array}\right\}$ . . . . $\left\{\begin{array}{l}x=12.\\y=9.\end{array}\right.$

6. ... $\left\{\begin{array}{l}\dfrac{x+y}{5}-\dfrac{x-y}{2}=3\\[2ex]\dfrac{x-y}{2}+\dfrac{x+y}{10}=0\end{array}\right\}$ . . . . $\left\{\begin{array}{l}x=4.\\y=6.\end{array}\right.$

As additional exercises, those under the two preceding methods may be taken.

SOLUTIONS AND CONDITIONS OF EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

263. Theorem I. If only one equation be given to determine two unknown quantities, their values are *indeterminate;* that is, they can have an indefinite number of values. *E.g.*,

$$\text{Let } x - y = a$$

be given to find the values of x and y that fulfil it. It is evident that if any value, as b, a known number, be given to y, the corresponding value of x will be found from the equation

$$x - b = a.$$

$$\text{Let } a = 10, \text{ then } x - y = 10;$$

and if any arbitrary values are assigned to y, the corresponding values of x will be found; thus,

$$\begin{array}{lll} \text{when } y = 1, & \text{then is} & x = 11, \\ \quad \ldots \ y = 2 & \ldots & x = 12, \\ \quad \ldots \ y = 3 & \ldots & x = 13, \\ \quad \&c. & & \&c. \end{array}$$

Any two of these corresponding values or systems of values will evidently fulfil the conditions of the equation. Thus, taking the system $y = 2$, $x = 12$, the equation becomes

$$12 - 2 = 10,$$

which being an identity, the equation is verified by these values and it is evident that it would be satisfied by any other system of values of x and y.

One equation containing two unknown quantities may therefore be satisfied by an indefinite number of pairs or systems of values of the two unknown quantities; but, in order that their values may be restricted to a single value of each, or one system of values, another equation must be added.

In order to prove that two equations will be sufficient to determine one system of values, provided they be consistent and independent, let the equations

$$ax + by = p \quad \ldots \quad [1]$$

$$cx + dy = q \quad \ldots \quad [2]$$

be given in which a, b, c, d, p, and q, are six known numbers.

By [1], $y = \dfrac{p - ax}{b}$ ... [3],

... [2], $y = \dfrac{q - cx}{d}$ ... [4],

and hence $\dfrac{p - ax}{b} = \dfrac{q - cx}{d}$ ... [5];

multiplying by bd, $dp - adx = bq - bcx$,

or $adx - bcx = dp - bq$

$(ad - bc)x = dp - bq$,

$\therefore x = \dfrac{dp - bq}{ad - bc}$ ... [6]

Substituting this value of x in one of the given equations, as in [1],

$$\frac{adp - abq}{ad - bc} + by = p,$$

$$by = p - \frac{adp - abq}{ad - bc} = \frac{adp - bcp - adp + abq}{ad - bc}$$

$$= \frac{abq - bcp}{ad - bc};$$

dividing by b, $y = \dfrac{aq - cp}{ad - bc}$.

These values have been found by the first method. In order to determine them by the second, let a value of y from one of the given equations, as from [2], be found

$$y = \frac{q - cx}{d},$$

and substituting this value for y in [1], it becomes

$$ax + \frac{bq - bcx}{d} = p,$$

or $adx + bq - bcx = dp$,

from which $(ad - bc)x = dp - bq$,

$$\therefore x = \frac{dp - bq}{ad - bc},$$

which is exactly the same as the value formerly found for x; and

it being substituted in [1] or [2], will give the same value for y as was formerly found.

To determine these values of x and y by the third method, multiply [1] by d and [2] by b, in order to eliminate y, then

$$adx + bdy = dp,$$

$$bcx + bdy = bq;$$

hence $$(ad - bc)x = dp - bq,$$

$$\therefore x = \frac{dp - bq}{ad - bc},$$

which is the same value of x as that found above; and the corresponding value of y may be found by substituting this value of x in either of the given equations.

The third method, by equalising the coefficients, is generally the best; for when the given equations are previously cleared of fractions, this method preserves all the terms of an integral form, whereas the other two methods generally introduce fractional coefficients.

264. Theorem II. The two given equations can afford only one value for each of the two unknown quantities; that is, only one system of values.

Let x', y', be one system of values that satisfy them, and, if possible, let x'', y'', be a different system; then, by substituting these values, the equations become

$$ax' + by' = p \quad \ldots \quad [7],$$

$$cx' + dy' = q \quad \ldots \quad [8],$$

$$ax'' + by'' = p \quad \ldots \quad [9],$$

and $$cx'' + by'' = q \quad \ldots \quad [10].$$

Taking the difference between [7] and [9], and between [8] and [10], the results are

$$a(x' - x'') + b(y' - y'') = 0 \quad \ldots \quad [11],$$

$$c(x' - x'') + d(y' - y'') = 0 \quad \ldots \quad [12].$$

Multiplying [11] by d, and [12] by b, and subtracting the products, the remainder is

$$ad(x' - x'') - bc(x' - x'') = 0,$$

or $$(ad - bc)(x' - x'') = 0;$$

dividing by $(ad - bc)$, then $$x' - x'' = 0 \quad \ldots \quad [13];$$

substituting this value of $x' - x''$ in [11], the first term becomes 0, and hence

$$b(y' - y'') = 0,$$

$$\therefore y' - y'' = 0 \quad \ldots \quad [14].$$

By [13] and [14], it appears that $x' = x''$, and $y' = y''$, or the second system of values, must be identical with the first, or, in other words, there can be but one system of values.

When the two given equations are dependent, the values of the two unknown quantities will be indeterminate. For if the equations are

$$ax + by = c,$$

and

$$5ax + 5by = 5c,$$

the latter is deducible from the former by multiplying by 5; and the latter being divided by 5, gives

$$ax + by = c,$$

which is the given equation, and nothing more can be derived from the two equations than from one of them. If any of the three methods of solution be applied to it, the result would be either nothing, or an identical equation, from which no particular value can be deduced (255).

Were the equations inconsistent, as

$$ax + by = c$$

$$ax + by = d,$$

where c, d, are supposed to be of *different* values; then, taking their difference, $0 = c - d$, or $c = d$; that is, they are also *equal.* Inconsistency, therefore, leads to *absurdity.*

265. When the number of given equations exceeds the number of unknown quantities, they will lead to absurd conclusions, unless some of them be dependent. The number of equations deducible from others—that is, dependent upon them—must be exactly equal to the excess of the number of the equations above that of the unknown quantities, in order that the latter may have determinate values. Thus, if the number of unknown quantities be two, and of equations five, then three of the latter must be dependent on some of the other equations; and these three being unnecessary, and therefore rejected, the remaining two will determine the values of the two unknown quantities.

266. Theorem III. When the number of given equations exceeds that of unknown quantities, if the latter be eliminated, the result will be equations containing only given quantities.

If these equations be consistent, it is a proof that some of the given equations are dependent, and therefore unnecessary; if they be inconsistent, this implies an impossibility of fulfilling the given equations by any values whatever of the unknown quantities; but if the given quantities are *disposable*—that is, if it be allowed to change their values arbitrarily—then these resulting equations may be all fulfilled by assigning proper values to these quantities. These values being properly adjusted, the said equations will be the *conditions* that must be fulfilled, in order that all the given equations be consistent, and the values of the unknown quantities determinate. Hence these equations among the given quantities are called *equations of condition.*

Let there be given two equations containing only one unknown quantity thus,

$$ax = b \quad \dots \quad [1],$$

$$cx = d \quad \dots \quad [2].$$

The value of x from the first, or $x = \frac{b}{a}$, being substituted in the second, gives

$$\frac{bc}{a} = d,$$

$$\text{or } bc = ad \quad \dots \quad [3];$$

and x being thus eliminated, the last equation among the given quantities is the equation of condition that they must fulfil, so that a value of x may be capable of satisfying the given equations. That condition $ad = bc$ may also be expressed thus (357),

$$a : b = c : d,$$

or these four quantities must be proportional. If they are not, the equation [3] cannot exist, and therefore implies an absurdity.

If there be given three equations containing only two unknown quantities, the latter being eliminated, the resulting equation among the known quantities will be the equation of condition. A value of the two unknown quantities being found from two of the given equations, these quantities may be eliminated from the other, by substituting these values for them. Thus, let the three equations

$$ax + by = p$$

$$cx + dy = q$$

$$mx + ny = r$$

be given, to find the equation of condition.

The values of x and y may be found from the first two, as in (263), which are

$$x = \frac{dp - bq}{ad - bc},\ y = \frac{aq - cp}{ad - bc},$$

and these values being substituted in the third, give

$$\frac{m(dp - bq)}{ad - bc} + \frac{n(aq - cp)}{ad - bc} = r,$$

or $$m(dp - bq) + n(aq - cp) + r(bc - ad) = 0,$$

and unless the values of the given quantities be such as to verify this last equation, or to make its first member $= 0$, no values of the unknown quantities can be found to satisfy the three equations. If, however, this equation be fulfilled, the values of x and y will then satisfy the three equations; but these values being found from the first two, the third is unnecessary.

EXERCISES WITH LITERAL COEFFICIENTS.

1. If $\left\{\begin{array}{l} ax - by = p \\ cx - dy = q \end{array}\right\}$. $x = \dfrac{bq - dp}{bc - ad},\ y = \dfrac{aq - cp}{bc - ad}.$

2. ... $\left\{\begin{array}{l} a(x - y) = p \\ b(x + y) = q \end{array}\right\}$ . . $x = \dfrac{aq + bp}{2ab},\ y = \dfrac{aq - bp}{2ab}.$

3. ... $\left\{\begin{array}{l} ax = by \\ x + y = c \end{array}\right\}$ . . $x = \dfrac{bc}{a + b},\ y = \dfrac{ac}{a + b}.$

4. ... $\left\{\begin{array}{l} \dfrac{m}{x} + \dfrac{n}{y} = a \\ \dfrac{n}{x} + \dfrac{m}{y} = b \end{array}\right\}$ . . $x = \dfrac{m^2 - n^2}{ma - nb},\ y = \dfrac{m^2 - n^2}{mb - na}.$

5. ... $\left\{\begin{array}{l} x + y = p \\ ax + by = q \end{array}\right\}$ . . $x = \dfrac{q - bp}{a - b},\ y = \dfrac{ap - q}{a - b}.$

6. ... $\left\{\begin{array}{l} \dfrac{x}{a} + \dfrac{y}{b} = p \\ \dfrac{x}{c} - \dfrac{y}{d} = q \end{array}\right\}$ $x = \dfrac{ac(bp + dq)}{ad + bc},\ y = \dfrac{bd(ap - cq)}{ad + bc}.$

EXAMPLES PRODUCING SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

1. The sum of two numbers is $= 60$, and their difference is $= 12$: what are these numbers?

$$\text{Let } x = \text{the greater,}$$
$$\text{and } y = \ldots \text{ less,}$$

then $\quad x + y = 60$, by the first condition,

and $\quad x - y = 12 \quad \ldots \quad$ second $\quad \ldots$

Taking the sum, $\quad 2x = 72$,

$$\therefore x = 36 \text{ the greater.}$$

Taking the difference, $\quad 2y = 48$,

$$\therefore y = 24 \text{ the less.}$$

These numbers verify the conditions of the question, for

$$x + y = 36 + 24 = 60, \text{ the sum,}$$
$$x - y = 36 - 24 = 12, \ldots \text{ difference.}$$

2. The sum of two numbers is $= 44$, and their ratio is that of 5 to 6: required the numbers.

$$\text{Let } x = \text{the greater,}$$
$$\text{and } y = \ldots \text{ less,}$$

then, by the conditions of the question,

$$x + y = 44 \quad \ldots \quad [1],$$

and $\quad x : y = 6 : 5$;

therefore $\quad 5x = 6y \quad \ldots \quad [2]$.

By [1] $\quad y = 44 - x$,

substituting this value of y in [2],

$$5x = 6(44 - x) = 264 - 6x$$
$$11x = 264,$$
$$\therefore x = 24;$$

hence $\quad y = 44 - x = 44 - 24 = 20$.

3. The sum of two numbers is $= 16$, and the sum of their reciprocals is $=$ double the difference of their reciprocals: what are the numbers?

Let $x =$ the less,

and $y =$ the greater,

then $x + y = 16 \quad \ldots \quad [1]$,

and $\dfrac{1}{x} + \dfrac{1}{y} = 2\left(\dfrac{1}{x} - \dfrac{1}{y}\right) \ldots [2]$;

multiply [2] by xy, $y + x = 2y - 2x$,

$\therefore x = \dfrac{y}{3} \quad \ldots \quad [3]$;

substituting this value of x in [1], $\dfrac{y}{3} + y = 16$

$$y + 3y = 48$$

$$4y = 48,$$

$$\therefore y = 12,$$

and by [3] $x = \dfrac{y}{3} = 4.$

4. The sum of two numbers is $= 10$, and twice the less is to three times the greater, as the square of the less to that of the greater: required the numbers.

Let $x =$ the less,

and $y =$ the greater,

then $x + y = 10 \quad \ldots \quad [1]$,

and $2x : 3y = x^2 : y^2 \ldots \quad [2]$.

By [2], $2xy^2 = 3x^2y \quad \ldots \quad [3]$,

dividing by xy, $2y = 3x \quad \ldots \quad [4]$,

by [1] $2x + 2y = 20$,

substituting for $2y$, $2x + 3x = 20$,

$\therefore x = 4$,

and by [4] $y = \dfrac{3x}{2} = \dfrac{12}{2} = 6.$

Equation [3] is a cubic equation, but it is easily reduced to a simple one by dividing by xy.

5. Divide a line of 20 inches into two parts, whose ratio shall be that of 8 to 12, or $2 : 3$.

$$\text{Let } x = \text{the shorter part,}$$

and $y = \text{the longer} \quad \ldots \; ;$

then $x + y = 20 \quad \ldots \quad [1],$

and $x : y = 2 : 3\,;$

or $3x = 2y,$

by [1], $3x + 3y = 60,$

taking the difference of the last two,

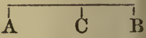
$$3y = 60 - 2y$$

$$5y = 60,$$

$$\therefore\ y = 12\,;$$

by [1], $x = 20 - y = 20 - 12 = 8.$

6. To divide a line a (AB) into two parts that shall have to each other the ratio of two numbers m and n.



Let $AB = a$ the given line,
$x = \text{the greater part} = AC,$

and $y = \text{the less} \quad \ldots \; = BC,$

then if $m > n$, $x : y = m : n,$

or $nx = my \quad \ldots \quad [1]\,;$

also $x + y = a \quad \ldots \quad [2],$

$$x = a - y$$

$$mx = am - my \quad \ldots \quad [3],$$

adding [1] and [3],

$$(m + n)x = am,$$

$$\therefore\ x = \frac{am}{m + n},$$

by [2], $y = a - x = a - \dfrac{am}{m + n} = \dfrac{an}{m + n}.$

This question is the same in principle as the first case in (256). This example is also a simple illustration of the application of algebra to geometry. The length a of the line may be expressed in inches or in any other denomination, observing that the values of x and y are expressed in the same denomination.

7. To produce a given line AB to a point as C, so that AC shall be to CB as m to n.

Let $a = \text{AB}$, the given line

$x = \text{AC}$,

and $y = \text{BC}$,

then $x : y = m : n$,

or $nx = my$ ... [1],

and, in this case, $x - y = a$ ... [2],

for AB is the difference between AC and BC;

then, by [2], $x = y + a$

$mx = my + am$ ... [3];

subtracting [1] from [3] $mx - nx = am$,

$$\therefore x = \frac{am}{m - n},$$

by [1], $$y = x - a = \frac{am}{m - n} - a = \frac{an}{m - n}.$$

Were $m = n$, x and y are each $= \infty$, or C is at an infinite distance, or, in other words, the problem is impossible.

Were $m \angle n$, x and y are negative, and in the ordinary sense of a negative quantity the question is impossible; but by the conventional meaning of a negative quantity it would then appear that the point C lies in an opposite direction, as C′, so that AC′ would be $= x$, and BC′ $= y$.

But x was supposed to be the greater; and it might be concluded, therefore, that the equation

$$x - y = a$$

would be inconsistent with this supposition. But it becomes in this case

$$-x - (-y) = -x + y = y - x = a,$$

and is the same as the difference between x and y considered as positive; and this results from the principle, that to subtract a quantity is to add it with an opposite sign.

8. The values of the properties belonging to two persons are as m to n, and the difference of their values is $= a$: required the values of the respective properties.

Let $x =$ the value of the greater property,

and $y =$ ... smaller ... .

Then as $a =$ the difference of the properties, the solution will lead to the same result exactly as in the last question; therefore

$$x = \frac{am}{m - n}, \text{ and } y = \frac{an}{m - n}.$$

If while x is the greater, m should be $\angle n$, then x and y are negative, and the negative value of x is less than that of y (55). Instead of x and y then representing absolute property, they represent only negative property, or debt. If an amount of property $= a$ be added to the negative property y, the sum will be equal to the negative property x, for

$$y + a = \frac{an}{m - n} + a = \frac{am}{m - n} = x$$

where x and y are negative, because $(m - n)$ is so, since $m \angle n$.

9. A's age is double of B's, but 15 years ago A's age was 5 times B's: required their ages.

Let $x =$ A's age,

and $y =$ B's ...,

then $x = 2y$ ... [1],

and $x - 15 = 5(y - 15) = 5y - 75$ ... [2].

By [2], $x = 5y - 60$,

subtracting [1], $0 = 3y - 60$,

or $y = 20$,

and $x = 2y = 40$.

10. A bill of £210 was paid in sovereigns and crowns (each = 5s.); and the number of the latter used was three times that of the former: how many pieces of each were necessary?

Let $x =$ the number of sovereigns,

and $y =$ ... ... crowns,

then $y = 3x$,

and $20x + 5y = 210 \times 20 = 4200$,

substituting the former value of y in the latter equation,

$$20x + 15x = 4200$$

$$35x = 4200,$$

$$\therefore x = 120,$$

and $y = 3x = 360$.

11. There is a certain fraction, to the numerator of which if 2 be added, the fraction becomes $= \frac{7}{8}$; and if from its denominator 2 be subtracted, it becomes equal to $\frac{5}{6}$: what is the fraction?

Let $x =$ the numerator of the fraction,

and $y = \ldots$ denominator $\ldots$;

then $$\frac{x+2}{y} = \frac{7}{8},$$

and $$\frac{x}{y-2} = \frac{5}{6}.$$

Multiply the former equation by $8y$, and the latter by $6(y-2)$, and they become

$$8x + 16 = 7y,$$

and $$6x = 5y - 10.$$

From the first member of the former, transpose 16 to the other side; then multiply the former by 5, and the latter by 7, and they become

$$40x = 35y - 80,$$

and $$42x = 35y - 70.$$

Taking the difference of these,

$$2x = 10,$$

$$\therefore x = 5;$$

and by the third equation, $7y = 8x + 16 = 40 + 16 = 56,$

$$\therefore y = 8.$$

Hence the fraction is $\frac{5}{8}$.

By numbering the equations [1], [2], [3], &c., and using the signs $+$, $-$, $\times$, $\div$, the various operations may be more concisely expressed. The preceding steps of solution will then be represented thus:—

$$\frac{x+2}{y} = \frac{7}{8} \quad \ldots \quad [1],$$

$$\frac{x}{y-2} = \frac{5}{6} \quad \ldots \quad [2],$$

[1] $\times 8y$, $\quad 8x + 16 = 7y,$

$$8x = 7y - 16 \quad \ldots \quad [3],$$

[2] $\times 6(y-2)$, $\quad 6x = 5y - 10 \quad \ldots \quad [4],$

[3] $\times 5$, $\quad 40x = 35y - 80 \quad \ldots \quad [5],$

[4] $\times 7$, $\quad 42x = 35y - 70 \quad \ldots \quad [6],$

[6] $-$ [5], $\quad 2x = 10,$

$$\therefore x = 5;$$

by [4], $5y = 6x + 10 = 30 + 10 = 40,$

$\therefore y = 8,$

and the fraction is $= \dfrac{x}{y} = \dfrac{5}{8}.$

12. If from the double of a certain number 6 be subtracted, the remainder will be a number whose digits are those of the former in an inverted order, and the sum of the digits is $= 6$: required the number.

Let $x =$ digit in the place of tens,

and $y =$ ... ... units,

then $10x + y =$ the number (443);

and therefore by the conditions of the question,

$$2(10x + y) - 6 = 10y + x \quad \ldots \quad [1],$$

also $x + y = 6$ ... ... [2],

by the former, $20x + 2y - 6 = 10y + x$

$$19x = 8y + 6,$$

by [2], $8x = -8y + 48,$

adding the last two, $27x = 54,$

$\therefore x = 2,$

by [2], $y = 6 - x = 6 - 2 = 4;$

and therefore the number is $= 10 \times 2 + 4 = 24.$

13. The sum of two numbers $= s$, and their difference $= d$: required the numbers.

Let $x =$ the greater,

and $y =$ the less;

then $x + y = s,$

and $x - y = d,$

adding $2x = s + d$, and $x = \dfrac{1}{2}(s + d),$

taking the difference $2y = s - d$, and $y = \dfrac{1}{2}(s - d).$

This question has been solved in another manner in the example preceding art. (249).

EXERCISES.

1. The sum of two numbers is = 30, and their difference = 6: what are the numbers? . . . . . = 12 and 18.

2. The sum of two numbers is = 60, and their ratio = that of 2 to 3: required the numbers, ... = 24 and 36.

3. The difference between two numbers is = 8, and twice the sum of their reciprocals is = three times the difference of their reciprocals: required the numbers, . . . = 2 and 10.

4. The sum of two numbers is = 14, and 3 times the less is to 4 times the greater, as the square of the less to the square of the greater: what are the numbers? . . . . = 6 and 8.

5. Divide a line of 36 inches into two parts, whose ratio shall be that of 5 to 7, ... = 15 and 21.

6. Find two numbers such that half the first with a third of the second shall be = 9, and a fourth part of the first with a fifth part of the second shall be = 5, ... = 8 and 15.

7. Two purses together contain 300 sovereigns, and if 30 sovereigns are taken out of the one purse and put into the other, there will be the same number in each: how many sovereigns are contained in each purse? = 180 in the one, 120 in the other.

8. The ages of two persons are in the ratio of 3 to 4, but 10 years ago the ratio of their ages was that of 2 to 3: required their ages, ... = 30 and 40.

9. Seven years ago, the age of a person (A) was just 3 times that of another (B); and seven years hence A's age will be just double that of B's: what are their ages? . . = 49 and 21.

10. A person wished to distribute 3d. apiece to some poor persons, but found he had not money enough in his pocket by 8d.; he therefore gave them each 2d., and found he had 3d. remaining: required the number of poor people, and the money he had in his pocket, . . = 11 poor persons, and 25d.

11. A cask, which held 60 gallons, was filled with a mixture of brandy, wine, and cider in such proportions that the cider was 6 gallons more than the brandy, and the wine was as much as the cider and one-fifth of the brandy: how much was there of each? . . Brandy = 15; cider = 21; and wine = 24.

12. A said to B, if you give me 10 guineas of your money, I shall then have twice as much as you will have left; but B said to A, give me 10 of your guineas and then I shall have three times as many as you: how many had each? A = 22; B = 26.

13. A farmer wishing to purchase a number of sheep, found that if they cost him 20s. a head he would be £2 short of money; but were they to be only 16s., he would then have £1 over: how many sheep were there, and how much money had he?
Sheep = 15, and money = £13.

14. A person (A) departs from a certain place, and travels at the rate of 7 miles in 5 hours, and 8 hours after another person (B) sets out from the same place, and travels at the rate of 5 miles in 3 hours: how long and how many miles does the first travel before he is overtaken by the second? = 50 hours, and 70 miles.

15. There is a certain number which is equal to 7 times the digit in the place of units, and if 18 be added to it, the sum is a number whose digits are those of the given number in an inverted order: what is the number? . . . . . = 35.

16. A cistern can be filled with water by two pipes running together in 6 hours, and the quantities conveyed by the pipes in the same time are as 3 to 4: in what time would they separately fill the cistern? . . . . . . = 14 and $10\frac{1}{2}$ hours.

17. Find that fraction which if 1 be added to its numerator its value will be $\frac{1}{3}$, but if 1 be added to its denominator, its value will be $\frac{1}{4}$, ... $= \frac{4}{15}$.

18. A person had two casks, the larger of which he filled with ale, and the smaller with cider. Ale being half-a-crown, and cider 11s. per gallon, he paid £8, 6s.; but had he filled the larger with cider and the smaller with ale, he would have paid £11, 5s. 6d.: how many gallons did each contain?
The larger = 18, and the smaller = 11 gallons.

III.—SIMPLE EQUATIONS CONTAINING THREE UNKNOWN QUANTITIES.

267. By properly modifying them, the three methods given in the preceding case may be extended to this one; and a fourth method may sometimes also be advantageously used here.

I. BY EQUATING.

268. Rule. Find a value of one of the unknown quantities in each of the three equations, and equate these values, so as to form two equations, by either placing the first equal to the second, and likewise to the third; or by putting the second equal to the first, and also to the third; and as these equations will contain only two unknown quantities, their values may be found as in the former case.

The equations ought to be cleared of fractions, if necessary, before applying this rule.

EXAMPLE.

If $\left\{\begin{array}{r} 2x - 3y + 5z = 15 \\ 3x + 2y - z = 8 \\ -x + 5y + 2z = 21 \end{array}\right\}$ what is the value of x, y, and z?

By the first equation, $z = \dfrac{15 - 2x + 3y}{5}$ ... [4],

... second ... $z = 3x + 2y - 8$ ... [5],

... third ... $z = \dfrac{x - 5y + 21}{2}$ ... [6].

Equating [4] and [5], $\dfrac{15 - 2x + 3y}{5} = 3x + 2y - 8$,

and reducing this equation, $17x = 55 - 7y$ ... [7].

Equating [5] and [6], $3x + 2y - 8 = \dfrac{x - 5y + 21}{2}$,

and reducing $5x = 37 - 9y$ ... [8].

The equations [7] and [8], resulting from the process of equating, are two equations containing two unknown quantities, x and y, whose values may be found by the rules of the former case. These values are thus found to be

$$x = 2, y = 3,$$

and by [5], $z = 6 + 6 - 8 = 4$.

EXERCISES.

1. If $\left\{\begin{array}{r} 2x - y + z = 9 \\ x - 2y + 3z = 14 \\ 3x + 4y - 2z = 7 \end{array}\right\}$ . . . . $\left\{\begin{array}{l} x = 3. \\ y = 2. \\ z = 5. \end{array}\right.$

2. ... $\left\{\begin{array}{l} \dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{4} = 62 \\ \dfrac{x}{3} + \dfrac{y}{4} + \dfrac{z}{5} = 47 \\ \dfrac{x}{4} + \dfrac{y}{5} + \dfrac{z}{6} = 38 \end{array}\right\}$ . . . $\left\{\begin{array}{l} x = 24. \\ y = 60. \\ z = 120. \end{array}\right.$

II. BY SUBSTITUTION.

269. Rule. Find a value of one of the unknown quantities in one of the equations in terms of the other two unknown quantities, and substitute this value for it in the other two equations which will then contain only two unknown quantities; and their values being found by former rules, that of the first may then be easily found, from its former value in terms of the two that are found.

EXAMPLE.

Given the three equations,

$$x - 2y + 3z = 11 \quad \ldots \quad [1],$$

$$8x + 2y - 4z = 10 \quad \ldots \quad [2],$$

$$\frac{x}{3} - \frac{y}{5} + \frac{z}{2} = 3 \quad \ldots \quad [3],$$

to find the values of x, y, and z.

As x in [1] is least involved, find a value of it in terms of y and z, thus —

$$x = 11 + 2y - 3z \quad \ldots \quad [4].$$

Before substituting this value of x in the other two equations, they may first be simplified by dividing [2] by 2, and clearing [3] of fractions; they then become

$$4x + y - 2z = 5$$

$$10x - 6y + 15z = 90;$$

and substituting the former value of x in these, we have

$$44 + 8y - 12z + y - 2z = 5$$

$$110 + 20y - 30z - 6y + 15z = 90,$$

or these become respectively

$$9y - 14z = -39$$

$$14y - 15z = -20.$$

These two equations contain only two unknown quantities, whose values are found by former rules to be

$$y = 5, \; z = 6.$$

and substituting these in [4],

$$x = 11 + 10 - 18 = 3$$

EXERCISES.

1. If $\left\{\begin{array}{l} 3x + 2y + \ \ z = 23 \\ 5x + 2y + 4z = 46 \\ 10x + 5y + 4z = 75 \end{array}\right\}$ . . . . $\left\{\begin{array}{l} x = 4. \\ y = 3. \\ z = 5. \end{array}\right.$

2. ... $\left\{\begin{array}{l} \dfrac{x}{3} - \dfrac{y}{2} + \ z = 3 \\ \dfrac{x}{6} + \dfrac{y}{4} - \dfrac{z}{3} = 1 \\ \dfrac{x}{2} - \dfrac{y}{4} + \ z = 5 \end{array}\right\}$ . . . $\left\{\begin{array}{l} x = 6. \\ y = 4. \\ z = 3. \end{array}\right.$

III. BY EQUALISING COEFFICIENTS.

270. Rule. Equalise the coefficients of one of the unknown quantities in the three equations, as in the former case; then take the difference between any two of them; next that between one of these equations and the remaining equation; and there will result two equations containing two unknown quantities.

EXAMPLE.

Find x, y, and z, in the three equations,

$$3x - 4y + z = 14 \quad \ldots \quad [1],$$

$$\frac{x}{3} + \frac{y}{5} - \frac{z}{9} = 4 \quad \ldots \quad [2],$$

$$6x - 4y + 2z = 68 \quad \ldots \quad [3].$$

Multiplying [2] by 45, and dividing [3] by 2, they become

$$15x + 9y - 5z = 180 \quad \ldots \quad [4],$$

$$3x - 2y + z = 34 \quad \ldots \quad [5].$$

Again, multiplying [1] and [5] by 5, in order to equalise the coefficients of z, they give

$$15x - 20y + 5z = 70 \quad \ldots \quad [6],$$

$$15x - 10y + 5z = 170 \quad \ldots \quad [7].$$

The coefficients of z are now equal in [4], [6], and [7]. Taking the sum of the two former, and the difference between the two latter, there results

$$30x - 11y = 250$$

$$10y = 100.$$

In taking the difference between [6] and [7], it is found that x vanishes as well as z; the cause of this elimination of x is—that its coefficient happened to be the same in [1] and [5]. Had x not disappeared, the last two equations obtained would have contained two unknown quantities. By the latter,

$$y = 10,$$

and substituting this in the former,

$$30x - 110 = 250,$$
$$\therefore x = 12.$$

Substituting now these values of x and y in one of the given equations, as, for instance in the first,

$$36 - 40 + z = 14,$$
$$\therefore z = 18.$$

EXERCISES.

1. If $\left\{\begin{aligned} 2x - 4y + 9z &= 28 \\ 7x + 3y - 5z &= 3 \\ 9x + 10y - 11z &= 4 \end{aligned}\right\}$ . . . $\left\{\begin{aligned} x &= 2. \\ y &= 3. \\ z &= 4. \end{aligned}\right.$

2\. ... $\left\{\begin{aligned} \frac{x}{3} - \frac{y}{4} + \frac{z}{2} &= 5 \\ \frac{x}{9} + \frac{y}{3} - \frac{z}{5} &= 3 \\ 2x - \frac{y}{6} + \frac{z}{10} &= 17 \end{aligned}\right\}$ . . . $\left\{\begin{aligned} x &= 9. \\ y &= 12. \\ z &= 10. \end{aligned}\right.$

3\. ... $\left\{\begin{aligned} x + 2y + 3z &= 17 \\ 2x + 3y + z &= 12 \\ 3x + y + 2z &= 13 \end{aligned}\right\}$ . . . $\left\{\begin{aligned} x &= 1. \\ y &= 2. \\ z &= 4. \end{aligned}\right.$

4\. ... $\left\{\begin{aligned} \frac{1}{x} + \frac{1}{y} &= a \\ \frac{1}{x} + \frac{1}{z} &= b \\ \frac{1}{y} + \frac{1}{z} &= c \end{aligned}\right\}$ . . . . $\left\{\begin{aligned} x &= \frac{2}{a + b - c}. \\ y &= \frac{2}{a + c - b}. \\ z &= \frac{2}{b + c - a}. \end{aligned}\right.$

As this is in general the best rule for the solution of equations with three unknown quantities, the reader should also now solve the exercises given under the two preceding cases by it.

IV. BY CONDITIONAL MULTIPLIERS.*

271. RULE. Multiply any two of the equations by some undetermined quantities, as m and n respectively, and from the sum of the resulting equations subtract the remaining equation; the remainder will be an equation containing the three unknown quantities. Any two of the unknown quantities may then be made to vanish, by assuming their coefficients equal to zero; and the value of the remaining unknown quantity may be found in terms of m and n. The quantities m and n may now have their values determined by means of the two assumed equations; and the three unknown quantities will then be easily found.

EXAMPLE.

If $\left\{\begin{array}{l} 2x + 3y - 4z = 10 \\ 3x - 2y + 5z = 11 \\ 4x + 5y - 2z = 28 \end{array}\right\}$ what are the values of x, y, and z?

Multiplying the first equation by m, and the second by n,

$$2mx + 3my - 4mz = 10m$$
$$3nx - 2ny + 5nz = 11n.$$

Subtracting the third from the sum of these,

$$(2m + 3n - 4)x + (3m - 2n - 5)y - (4m - 5n - 2)z = 10m + 11n - 28 \quad \ldots \quad [1].$$

Assuming the coefficients of x and y equal to zero,

$$2m + 3n - 4 = 0 \quad \ldots \quad [2],$$
$$3m - 2n - 5 = 0 \quad \ldots \quad [3];$$

hence $$-(4m - 5n - 2)z = 10m + 11n - 28,$$

$$\therefore z = -\frac{10m + 11n - 28}{4m - 5n - 2}.$$

From the two assumed equations containing m and n, their values are found (by the rule of the former case) to be

$$m = \frac{23}{13}, \text{ and } n = \frac{2}{13};$$

hence substituting these values of m, n, in that of z, we find

$$z = 2.$$

* See Lacroix, *Elémens d'Algèbre*, seizième edition, p. 131; and Garnier, *Élémens d'Algèbre*, troisième edition, p. 216.

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In order to find the value of y, the coefficients of x and z in [1] must now be assumed equal to zero, or

$$2m + 3n - 4 = 0 \quad \ldots \quad [4],$$

$$4m - 5n - 2 = 0 \quad \ldots \quad [5];$$

then $$y = \frac{10m + 11n - 28}{3m - 2n - 5},$$

and the new values of m and n may be found from these two assumed equations [4] and [5], and being substituted in that of y, give its value.

The value of x is to be found in a similar manner, by assuming the coefficients of y and z in [1] equal to zero; or it may be found from one of the given equations, by substituting in it the values found of y and z.

It is left as an exercise to find these values of x and y, which are respectively $= 3$ and 4.

ELIMINATION BY CROSS MULTIPLICATION.

272. Let the three general equations of the first degree containing three unknown quantities be given—namely,

$$ax + by + cz = d$$

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2.$$

It is required to determine three conditional multipliers, l, m, n, such that if the first equation be multiplied by l, the second by m, and the third by n, and the three resulting equations added together, the coefficients of two of the unknown quantities shall each be $=$ zero.

Multiplying each of the above equations as proposed, we have the following results:—

$$lax + lby + lcz = ld$$

$$ma_1x + mb_1y + mc_1z = md_1$$

$$na_2x + nb_2y + nc_2z = nd_2.$$

Since three conditional multipliers have been introduced, they may be determined so as to fulfil three conditions: let the first two conditions be that when the equations are added, the coefficients of y and z shall each $= 0$; then the resulting equation will be

$$(la + ma_1 + na_2)x = ld + md_1 + nd_2,$$

$$\therefore x = \frac{ld + md_1 + nd_2}{la + ma_1 + na_2}.$$

Again, since the coefficients of y and z are each $=$ zero,

$$\left.\begin{array}{l} lb + mb_1 + nb_2 = 0 \\ lc + mc_1 + nc_2 = 0 \end{array}\right\} \text{ or } \left\{\begin{array}{l} lbc + mb_1c + nb_2c = 0 \\ lbc + mbc_1 + nbc_2 = 0, \end{array}\right.$$

by multiplying the first of these equations by c, and the second by b, and, subtracting the results and transposing, we obtain

$$(bc_1 - b_1c)m = (b_2c - bc_2)n;$$

hence
$$\frac{m}{b_2c - bc_2} = \frac{n}{bc_1 - b_1c}.$$

Also by multiplying the first by c_1, and the second by b_1, and subtracting the results and transposing, we obtain

$$(bc_1 - b_1c)l = (b_1c_2 - b_2c_1)n;$$

hence
$$\frac{l}{(b_1c_2 - b_2c_1)} = \frac{n}{bc_1 - b_1c},$$

$$\therefore \frac{l}{b_1c_2 - b_2c_1} = \frac{m}{b_2c - bc_2} = \frac{n}{bc_1 - b_1c}.$$

Thus it appears that only the ratio of the multipliers has been determined, and therefore there is an indefinite number of multipliers that will satisfy the conditions: but to obtain the simplest forms of these multipliers, let the third condition be that each of the above ratios $=$ unity; then to find x,

$$\left.\begin{array}{l} l = b_1c_2 - b_2c_1 \\ m = b_2c - bc_2 \\ n = bc_1 - b_1c \end{array}\right\} \quad (A);$$

$$\therefore x = \frac{d(b_1c_2 - b_2c_1) + d_1(b_2c - bc_2) + d_2(bc_1 - b_1c)}{a(b_1c_2 - b_2c_1) + a_1(b_2c - bc_2) + a_2(bc_1 - b_1c)}.$$

Similarly
$$y = \frac{d(a_1c_2 - a_2c_1) + d_1(a_2c - ac_2) + d_2(ac_1 - a_1c)}{b(a_1c_2 - a_2c_1) + b_1(a_2c - ac_2) + b_2(ac_1 - a_1c)},$$

and
$$z = \frac{d(a_1b_2 - a_2b_1) + d_1(a_2b - ab_2) + d_2(ab_1 - a_1b)}{c(a_1b_2 - a_2b_1) + c_1(a_2b - ab_2) + c_2(ab_1 - a_1b)}.$$

In order to solve any equation with three unknown quantities, it is therefore only necessary to find the values of l, m, and n from (A), to multiply the successive equations by these values, and to add the results to obtain a simple equation containing only x, and known quantities from which its value may be easily found. In the same manner multipliers can be found for eliminating x and z, so

as to find the value of y; or, for eliminating x and y, so as to find the value of z; these are

$$\left.\begin{aligned} l &= a_1c_2 - a_2c_1 \\ m &= a_2c - ac_2 \\ n &= ac_1 - a_1c \end{aligned}\right\} \text{(B) for finding } y.$$

$$\left.\begin{aligned} l &= a_1b_2 - a_2b_1 \\ m &= a_2b - ab_2 \\ n &= ab_1 - a_1b \end{aligned}\right\} \text{(C) for finding } z.$$

Let the student now write the three given equations under each other, and the first a second time under the third; thus

$$\begin{aligned} ax + by + cz &= d \\ a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ ax + by + cz &= d. \end{aligned}$$

If the factors of the conditional multipliers, as given in (A), (B), and (C), be now traced out in the above form, a very simple and symmetrical mode of forming them will be discovered, which may be applied at once to any system of three equations, with three unknown quantities, whether the coefficients be literal or numerical.*

EXAMPLE.

$$\text{Given } \left\{\begin{aligned} 2x + 3y - 4z &= 10 \\ 3x - 2y + 5z &= 11 \\ 4x + 5y - 2z &= 28 \end{aligned}\right\} \text{ to find } x, y, \text{ and } z.$$

Here, to find x, $l = -2 \times -2 - 5 \times 5 = -21$, $m = 5 \times -4 - 3 \times -2 = -14$, and $n = 3 \times 5 - (-2) \times (-4) = 7$.

Multiplying by these, we find

$$\begin{aligned} -42x - 63y + 84z &= -210 \\ -42x + 28y - 70z &= -154 \\ 28x + 25y - 14z &= 196 \\ \hline \text{Sum} = \qquad\qquad -56x &= -168, \\ \therefore x &= 3. \end{aligned}$$

In the same manner the values of y and z may be found; the multipliers for finding y being, $l = -26$, $m = -12$, and $n = 22$ whilst those for finding z are, $l = 23$, $m = 2$, and $n = -13$.

* This method was first given in the *Cambridge Mathematical Journal*, vol. i page 46, under the name of Elimination by Cross Multiplication.

Scholium.—The method of equating is the simplest in principle, though not in practice; and it may be easily extended to any number of simple equations containing the same number of unknown quantities. Thus, for four such equations, find a value of one of the unknown quantities in each of the four equations; next equate these values in pairs, and three independent equations will be found, containing three unknown quantities, whose values are to be found as formerly; then the value of the fourth unknown quantity will be easily found.

Thus, the case of four equations with four unknown quantities is reduced to that of three; and in a similar manner, that of five is reduced to that of four; and by a similar process the solution of any number of equations may be reduced to that of a less number by unity.

This, however, is effected in a much simpler manner by using the method of conditional multipliers, the number of which must be one less than the number of equations. Whatever be the number of equations, if all of them but one be multiplied in order by the undetermined multipliers, and from the sum of these products the remaining equation be subtracted, the resulting equation will contain all the unknown quantities, and the coefficients of all of them but one being assumed equal to zero, there will remain only one unknown quantity, whose value is then easily found in terms of the multipliers. The values of the multipliers are then to be determined from the equations that were found by equating the coefficients with zero, the number of which is one less than that of the given equations; so that the solution of the given equations is now reduced to that of a number of equations less by unity. These multipliers being now considered as the unknown quantities, they may, all but one, be similarly eliminated, by multiplying the latter equations, except one, by new conditional multipliers; and the number of equations to be solved containing the new multipliers as the unknown quantities, are again less than the preceding number by unity.

Thus, if the number of equations originally given be n, then $(n - 1)$ multipliers are to be used, and $(n - 1)$ equations containing them will be formed. Then $(n - 2)$ of these $(n - 1)$ equations are to be multiplied by $(n - 2)$ new multipliers, and $(n - 2)$ equations will be formed containing them. Thus, the number of equations containing them is diminished by unity at each step, and at last three equations only will remain, the mode of solving which is already known.

QUESTIONS PRODUCING SIMPLE EQUATIONS CONTAINING THREE UNKNOWN QUANTITIES.

EXAMPLES.

1. To find three numbers, such that the first with half the other two, the second with one-third of the other two, and the third with one-fourth of the other two, shall be each $= 34$.

Let x, y, z, be three numbers in order; then the three conditions afford respectively the three equations

$$x + \tfrac{1}{2}(y + z) = 34,$$

$$y + \tfrac{1}{3}(x + z) = 34,$$

$$z + \tfrac{1}{4}(x + y) = 34.$$

Multiplying these equations respectively by 2, 3, and 4, in order to clear them of fractions, we obtain

$$2x + y + z = 68 \quad \ldots \quad [1],$$

$$3y + x + z = 102 \quad \ldots \quad [2],$$

$$4z + x + y = 136 \quad \ldots \quad [3].$$

By $[2] \times 2$, $\quad 6y + 2x + 2z = 204 \quad \ldots \quad [4]$,

... $[4] - [1]$, $\quad 5y + z = 136 \quad \ldots \quad [5]$,

... $[3] - [2]$, $\quad 3z - 2y = 34$,

... $[5] \times 3$, $\quad 15y + 3z = 408$;

taking the difference of these, $\quad 17y = 374$,

$$y = 22;$$

by [5], $\quad z = 136 - 5y = 136 - 110 = 26$;

... [2], $\quad x = 102 - 3y - z = 102 - 66 - 26 = 10$.

2. A person has three ingots composed of three different metals in different proportions. A pound of the first consists of 7 ounces of silver, 3 of copper, and 6 of tin; one of the second consists of 12 ounces of silver, 3 of copper, and 1 of tin; and one of the third of 4 ounces of silver, 7 of copper, and 5 of tin: how much of each of the ingots must be taken to form another of 1 pound weight consisting of 8 ounces of silver, $3\frac{3}{4}$ of copper, and $4\frac{1}{4}$ of tin?

Let x, y, z, be the number of ounces to be taken of the three ingots in order.

Then,

$16 : x = 7 : \frac{7x}{16}$, the ounces of silver in x ounces of the first,

$16 : y = 12 : \frac{12y}{16}$ ... ... y ... second,

$16 : z = 4 : \frac{4z}{16}$ ... ... z ... third.

Hence, since 8 is the number of ounces of silver in a pound of the ingot to be formed, the first equation is

$$\frac{7x}{16} + \frac{12y}{16} + \frac{4z}{16} = 8;$$

or, multiplying by 16,

$$7x + 12y + 4z = 128 \quad \ldots \quad [1];$$

and in a similar manner are formed the two following equations for the copper and the tin:—

$$3x + 3y + 7z = 60 \quad \ldots \quad [2],$$

$$6x + y + 5z = 68 \quad \ldots \quad [3].$$

The coefficient of y being the simplest in these equations, it will be most convenient to eliminate it.

Multiplying the second equation by 4, and taking the first equation from the product, there remains

$$5x + 24z = 112 \quad \ldots \quad [4].$$

Multiplying the third equation by 3, and taking the second from the product, the result is

$$15x + 8z = 144 \quad \ldots \quad [5].$$

Multiplying this equation by 3, and taking the preceding from it,

$$40x = 320,$$

$$\therefore x = 8.$$

Substituting this value of x in [5],

$$120 + 8z = 144,$$

$$\therefore z = \frac{24}{8} = 3.$$

Again, substituting these values of x and z in [3],

$$48 + y + 15 = 68,$$

$$\therefore y = 5.$$

Hence, to form a pound of the ingot, there must be taken of the first, 8 ounces; of the second, 5; and of the third, 3 ounces.

EXERCISES.

1. Find three numbers, such that the sum of the first and second shall be = 7, the sum of the first and third = 8, and the sum of the second and third = 9, . . . = 3, 4, and 5.

2. There are three numbers, such that the first, with half of the second, is = 14; the second, with a third part of the third, is = 18; and the third, with a fourth of the first, is = 20: required the numbers, ... = 8, 12, and 18.

3. A person purchased three jewels: the price of the first, with half the price of the other two, was £25; the price of the second, with a third of that of the first and third, was £26; and the price of the third, with half the price of the other two, was £29: required the price of each, . = £8, £18, and £16.

4. A, B, and C, have a certain number of crowns: A gives to B and C as much as they already have; then B doubles to A and C what they now have; and lastly, C doubles to A and B what they already have; they now find that each has 16 crowns: how many had each at first? . . A = 26, B = 14, and C = 8.

5. A certain number is composed of three digits: the sum of the digits is 11; the digit in the place of units is double that in the place of hundreds; and if 297 be added to the number, its digits are inverted: required the number, . . . = 326.

6. Three brothers purchased an estate for £50,000: the money possessed by the first brother was insufficient to pay for the whole by a half of the money possessed by the second; the second could have paid for the whole, if he had had, in addition to his own money, a third of that of the first; and the third alone could also have paid for it, if he had had, in addition to his own, a fourth of the money of the first brother: how much money had each? . . . . . . = £30,000, £40,000, £42,500.

7. Find a number consisting of three places, such that the sum of the extreme digits is double that of the middle one; and if it be divided by the sum of its digits, the quotient will be = 48; also, if 198 be subtracted from the number, the digits will be inverted, ... = 432.

8. Required the distances between a town A, and other three towns B, C, and D, knowing merely that if to half the distance between A and C there be added three-fourths of the from A to D, and also 31·4 furlongs, the sum is the between A and B; and if to the distance between A and

be added half that between A and D, and 9·5 furlongs, the sum will be = the distance of A from C; and if from the distance between A and C there be taken half the distance between A and B, and 158·5 furlongs, the remainder will be = the distance from A to D, ... = 523·7, 646·05, and 225·7.

9. A and B can perform a piece of work in 8 days, A and C together in 9 days, and B and C together in 10 days: in what time could each alone perform it? $= 14\frac{34}{49}$, $17\frac{23}{41}$, and $23\frac{7}{31}$ days.

10. A cistern is filled by three pipes, A, B, C: the pipes A and B together fill the cistern in 70 minutes; A and C together in 84 minutes; B and C together in 140 minutes: in what time will each pipe fill the cistern, and in what time will it be filled if all three pipes are open?

A = 105 minutes, B = 210 minutes, C = 420 minutes, and A B C together = 60 minutes.

11. A and B can perform a piece of work in a days, A and C together in b days, and B and C together in c days: in what time will each perform it alone?

$$A = \frac{2abc}{ac + bc - ab} \text{ days, } B = \frac{2abc}{ab + bc - ac}, \text{ and } C = \frac{2abc}{ab + ac - bc}.$$

QUADRATIC EQUATIONS.

273. A *quadratic* equation is one in which the highest power of the unknown quantity is its square (236). When it contains only the second power, it is called a *pure* quadratic; but if it contain the first and second powers, it is called an *adfected* quadratic.

Thus, $ax^2 = b$ is a pure quadratic; and $ax^2 + bx = c$ is an adfected quadratic.

Any pure or adfected quadratic may be easily reduced to these two forms; for in a pure quadratic all the terms containing x^2 may be collected together into one sum, and then the compound coefficient of x^2 may be assumed equal to a single quantity, as m, and the sum of the constant terms to another quantity n, whence the equation becomes

$$mx^2 = n, \text{ or } mx^2 - n = 0;$$

or if $r = -n$, $$mx^2 + r = 0.$$

So an adfected quadratic may be similarly reduced; for all the

terms containing x^2 may be reduced to one term mx^2, and thos containing x to one, as nx, and the constant terms to one, as p whence the equation is

$$mx^2 + nx = p, \text{ or } mx^2 + nx - p = 0;$$

or if $r = -p$,

$$mx^2 + nx + r = 0.$$

I.—PURE QUADRATICS.

274. Rule. To solve a pure quadratic, find first the value o the square of the unknown quantity in the same manner as the value of the unknown quantity itself was found in simple equa- tions containing one unknown quantity; and the square roo of its value will be the value required.

EXAMPLES.

1. Find the values of x in the equation $x^2 - 9 = 0$.

$$x^2 - 9 = 0,$$

$$x^2 = 9,$$

$$\therefore x = \sqrt{9} = +3 \text{ or } -3; \text{ that is, } x = \pm 3.$$

The value of x is either positive or negative; and the unknown quantity has two equal values with opposite signs.

2. Given $2x^2 - 12 = 13 - 3x^2$, to find the values of x.

$$2x^2 - 12 = 13 - 3x^2,$$

$$5x^2 = 25,$$

$$x^2 = 5,$$

$$\therefore x = \pm\sqrt{5}.$$

As $\sqrt{5}$ is an irrational quantity, its value cannot be assigned in integers (195); but by extracting the square root of 5, by arithmetic, to any required number of decimal places, an approxi- mate value of it may be found to any required degree of accuracy.

By substituting either $+\sqrt{5}$ or $-\sqrt{5}$ for x in the equation, it will be found to verify it. For the square of $+\sqrt{5}$ or of $-\sqrt{5}$ is $= 5$, and putting 5 therefore for x^2, the equation becomes

$$2 \times 5 - 12 = 13 - 3 \times 5,$$

or

$$10 - 12 = 13 - 15, \text{ or } -2 = -2,$$

which is an identity (228).

3. Given $x^2 - a^2 = 0$.

$$x^2 = a^2,$$

$$\therefore x = +a \text{ or } -a; \text{ that is. }$$

To verify the equation for these two values of x, it becomes

$$(+a)^2 - a^2 = 0, \text{ or } a^2 - a^2 = 0,$$

nd

$$(-a)^2 - a^2 = 0, \text{ or } a^2 - a^2 = 0.$$

4. Given $a^2x^2 + b^2 = 0$.

$$a^2x^2 = -b^2,$$

$$x^2 = -\frac{b^2}{a^2} \text{ or } = \frac{b^2}{a^2}(-1),$$

$$\therefore x = \pm\sqrt{\frac{b^2}{a^2}}\sqrt{(-1)} = \pm\frac{b}{a}\sqrt{-1}.$$

These values of x are imaginary, as the factor $\sqrt{-1}$ is so; but they both satisfy the conditions of the equation, which, by substituting $+\frac{b}{a}\sqrt{-1}$ for x, becomes

$$a^2\left(\frac{b}{a}\sqrt{-1}\right)^2 + b^2 = 0,$$

or

$$a^2\frac{b^2}{a^2}(-1) + b^2 = 0,$$

$$-\frac{a^2b^2}{a^2} + b^2 = 0, \text{ or } -b^2 + b^2 = 0,$$

so that the equation is verified. If $-\frac{b}{a}\sqrt{-1}$ were substituted for x, it would also be found to fulfil the conditions of the equation. Since x^2 is positive, whether the value of x be positive or negative, the term a^2x^2 is always positive for a real value of x; and b^2 is positive; but the sum of two positive quantities cannot be $= 0$; and from this circumstance alone it appears that the equation is impossible for real values of x. When the root of such an equation therefore is imaginary, it proves the equation to be impossible for real values of the unknown quantity.

EXAMPLES.

1. If $x^2 - 8 = 28$, ... $x = \pm 6$.
2. ... $4x^2 - 64 = 0$, ... $x = \pm 4$.
3. ... $3x^2 - 15 = 83 + x^2$, ... $x = \pm 7$.
4. ... $a^2x^2 - b^2 = 0$, ... $x = \pm\frac{b}{a}$.

r if r = - ,

So an adfected quadı 0, ... $x = \pm\sqrt{\frac{b}{a}}\sqrt{-1}$.

6. If $d^2 + bx^2 = a^2$, ... $x = \pm\sqrt{\dfrac{a^2 - d^2}{b}}$.

7. ... $a^2x^2 + d^2 = c^2x^2 + b^2$, ... $x = \pm\sqrt{\dfrac{b^2 - d^2}{a^2 - c^2}}$.

8. ... $ax^2 + b = cx^2 + d$, . . . $x = \pm\left(\dfrac{d - b}{a - c}\right)^{\frac{1}{2}}$.

QUESTIONS PRODUCING PURE QUADRATIC EQUATIONS.

1. What number is that whose square being added to 50, the sum is = 99?

Let x = the number,

then $x^2 + 50 = 99$ by the question,

or $x^2 = 99 - 50 = 49$,

$\therefore\ x = \sqrt{49} = \pm\ 7.$

The number is therefore + 7 or − 7, the square of which is 49, and 49 + 50 = 99; so that the number satisfies the given condition.

2. Find a number such that 10 times its 5th power shall be equal to 250 times its cube.

Let x = the number,

then $10x^5 = 250x^3$,

or $x^2 = 25$,

$\therefore\ x = \sqrt{25} = \pm 5.$

3. Find a number such that $(m - 2)$ times its mth power shall be equal to m times its $(m + 2)$th power.

Let x = the number,

then $(m - 2)x^m = mx^{m+2}$,

or $mx^{m+2} = (m - 2)x^m$;

dividing by mx^m, $x^2 = \dfrac{m - 2}{m}$,

$$\therefore\ x = \pm\sqrt{\frac{m - 2}{m}}.$$

Since $m - 2 \angle m$, the value of x will be a fraction. If $m \angle 2$, the numerator is imaginary, and the question impossible.

4. A number is equal to 9 times its reciprocal: required the number.

a; th

Let $x =$ the number,

then $$x = 9 \times \frac{1}{x} = \frac{9}{x}, \text{ or } x^2 = 9,$$

$$\therefore x = \sqrt{9} = \pm 3.$$

5. If the height through which a heavy body falls in different periods of time be proportional to the squares of these times, in how many seconds will a body fall through 400 feet, the space it falls through in one second being $= 16{\cdot}1$ feet?

Let $x =$ the number of seconds occupied in falling through 400 feet;

then $$16{\cdot}1 : 400 = 1^2 : x^2,$$

hence $$16{\cdot}1x^2 = 400,$$

$$x^2 = \frac{400}{16{\cdot}1} = 24{\cdot}8,$$

$$\therefore x = \sqrt{24{\cdot}8} = \pm 4{\cdot}9, \text{ or nearly 5 seconds.}$$

The negative value $-4{\cdot}9$ implies that a heavy body thrown upwards with such a velocity as would make it ascend to 400 feet, would take 4·9 seconds of time to reach that height. The velocity cannot be taken in an opposite sense, but the direction of the motion can be so, and by this change a proper interpretation of the negative value is determined.

6. If the force of the earth's attraction at any point not within its surface diminish inversely as the square of its distance from the centre, at how many semi-diameters distant from the earth's centre will this force be 16 times less than at its surface?

Let the force at the earth's surface — that is, at the distance of a radius or semi-diameter from the centre — be called 1, and let x be the required distance—that is, the number of semi-diameters from the centre at which the force is 16 times less,

then $$1 : \frac{1}{16} = \frac{1}{1^2} : \frac{1}{x^2}, \text{ or } 16 : 1 = x^2 : 1;$$

$$\therefore x^2 = 16, \text{ and } x = \sqrt{16} = \pm 4 \text{ semi-diameters.}$$

EXERCISES.

1. Find a number such that the square of its half added to 6 shall $= 10$, ... $= \pm 4$.

2. Find a number such that if 3 times its cube be diminished by 5 times the number itself, the remainder shall be $= 7$ times the number, ... $= \pm 2$.

3. Find a number such that its fourth power shall be $= 9$ times its square, ... $= \pm 3$.

4. Find a number such that 10 times its sixth power shall be $= 40$ times its fourth power, ... $= \pm 2$

5. Find a number such that $(m - 2)$ times its $(n + 2)$th power shall be $= m$ times its nth power, . . $= \pm \sqrt{\frac{m}{m-2}}$

6. In what time will a body fall through a height of 1000 feet? (See 6th example), ... $= 7\cdot9$ seconds

7. If the attraction of a magnet, at the distance of 10 inches from its pole, can support a small weight of 20 grains, at what distance will it support only 4 grains, its attractive power varying nearly inversely as the square of the distance from its pole? . . . . $= 22\cdot36$ inches

II.—ADFECTED QUADRATICS CONTAINING ONLY ONE UNKNOWN QUANTITY.

275. Every adfected quadratic, as before observed (272), may be reduced to either of these two forms,

$$ax^2 + bx + r = 0,$$

or

$$ax^2 + bx = c;$$

and there will be the four following cases, according as b or c is positive or negative, namely—

$$ax^2 + bx = c \quad \ldots \quad (1),$$
$$ax^2 - bx = c \quad \ldots \quad (2),$$
$$ax^2 + bx = -c \quad \ldots \quad (3),$$
$$ax^2 - bx = -c \quad \ldots \quad (4),$$

which are all comprehended in the equation

$$ax^2 \pm bx = \pm c,$$

the signs being taken in any order.

276. When it happens in the first form that $4ar = (\pm b)^2 = b^2$ the equation is a complete square (180), and is of the form

$$m^2x^2 \pm 2mnx + n^2 = 0,$$

and its square root is (190) $= \ldots mx \pm n = 0$;

and hence

$$x = \mp \frac{n}{m}.$$

Thus, if $m = 2$ and $n = -3$, the equation is

$$4x^2 - 12x + 9 = 0,$$

and

$$4 \times 4 \times 9 = (-12)^2 = 144,$$

and its square root is

$$2x - 3 = 0,$$

$$\therefore x = \frac{3}{2}.$$

277. When the equation is a complete square, the value of x is thus easily found; but this seldom happens. When the equation is not a complete square, however, it may be solved by one of the two following methods:—

FIRST METHOD.

278. Rule. Arrange the equation, if necessary, so that the first member shall consist of two terms—the first term containing the square of the unknown quantity, and the second that quantity itself, observing that the second member contain only known quantities. Free the square of the unknown quantity of its coefficient, if it have any, by dividing the equation by it.

To make the first member a complete square, add to both sides of the equation the square of half the coefficient of the second term. Then extract the square root of both sides, and the result will be a simple equation, from which the value of the unknown quantity may easily be found by former rules.

Taking the equation

$$x^2 + 2ax = b^2,$$

complete the square by adding $\left(\frac{2a}{2}\right)^2$ or a^2 to both sides,

then

$$x^2 + 2ax + a^2 = a^2 + b^2.$$

Extracting the square root,

$$x + a = \pm \sqrt{(a^2 + b^2)},$$

or

$$x = -a \pm \sqrt{(a^2 + b^2)},$$

and x will have two values—one when the radical part is taken as positive, and another when negative. These two values are

$$x = -a + \sqrt{(a^2 + b^2)}$$
$$x = -a - \sqrt{(a^2 + b^2)}.$$

Thus, if the equation be

$$x^2 + 6x = 16,$$

then $2a = 6$ or $a = 3$, $b^2 = 16$, and the values of x are

$$x = -3 + \sqrt{(9 + 16)} = -3 + \sqrt{25} = -3 + 5 = 2$$
$$x = -3 - \sqrt{(9 + 16)} = -3 - \sqrt{25} = -3 - 5 = -8.$$

Or solving the equation according to the rule,

$$x^2 + 6x = 16,$$

completing the square by adding the half of 6 squared, or 3^2,

$$x^2 + 6x + 3^2 = 16 + 9 = 25.$$

Extracting the square root,

$$x + 3 = \pm\sqrt{25} = \pm 5$$

and $$x = -3 \pm 5 = 2 \text{ or } -8,$$

for when $+5$ is taken, $-3 + 5 = 2$; and for -5, $x = -3 - 5 = -8$.

EXAMPLES.

1. If $x^2 - 4x + 8 = 20$, what is the value of x?

Transposing, $$x^2 - 4x = 20 - 8 = 12,$$
completing the square, $$x^2 - 4x + 4 = 12 + 4 = 16,$$
extracting the square root, $$x - 2 = \pm\sqrt{16} = \pm 4,$$
$$\therefore\ x = 2 \pm 4 = 6 \text{ or } -2.$$

2. If $x^2 + 6x + 4 = 44$, what is the value of x?

$$x^2 + 6x = 44 - 4 = 40$$
$$x^2 + 6x + 9 = 40 + 9 = 49$$
$$x + 3 = \sqrt{49} = \pm 7,$$
$$\therefore\ x = \pm 7 - 3 = 4 \text{ or } -10.$$

3. If $x^2 + \frac{7}{2}x + 8 = 10$, what is the value of x?

$$x^2 + \frac{7}{2}x = 10 - 8 = 2.$$

Completing the square,

$$x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = 2 + \frac{49}{16} = \frac{81}{16}$$
$$x + \frac{7}{4} = \sqrt{\frac{81}{16}} = \pm\frac{9}{4},$$
$$\therefore\ x = \pm\frac{9}{4} - \frac{7}{4} = \frac{1}{2} \text{ or } -4.$$

4. If $3x^2 - 2x + 123 = 256$, what is the value of x?

$$3x^2 - 2x = 256 - 123 = 133,$$
$$\therefore\ x^2 - \frac{2}{3}x = \frac{133}{3}.$$

Completing the square,

$$x^2 - \frac{2}{3}x + \left(\frac{1}{3}\right)^2 = \frac{133}{3} + \frac{1}{9} = \frac{400}{9}.$$

Extracting, $$x - \frac{1}{3} = \sqrt{\frac{400}{9}} = \pm\frac{20}{3},$$
$$\therefore\ x = \frac{1}{3} \pm \frac{20}{3} = 7 \text{ or } -\frac{19}{3}.$$

5. If $3x^4 + 6x^3 = 45x^2$, what is the value of x?

Dividing by $3x^2$,

$$x^2 + 2x = 15$$
$$x^2 + 2x + 1 = 16$$
$$x + 1 = \sqrt{16} = \pm 4,$$
$$\therefore x = \pm 4 - 1 = 3 \text{ or } -5.$$

6. If $x^3 - 4x^2 + 5x = x^3 - 6x^2 + 18$, what is the value of x?

Cancelling x^3 from both sides,

$$-4x^2 + 5x = -6x^2 + 18$$
$$2x^2 + 5x = 18$$
$$x^2 + \frac{5}{2}x = 9$$
$$x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = 9 + \frac{25}{16} = \frac{169}{16}$$
$$x + \frac{5}{4} = \sqrt{\frac{169}{16}} = \pm \frac{13}{4},$$
$$\therefore x = \pm \frac{13}{4} - \frac{5}{4} = 2 \text{ or } -\frac{9}{2}.$$

7. If $\sqrt{(2x^2 - 2)} = 3x - 5$, what is the value of x?

Clearing the equation of radicals (246),

$$2x^2 - 2 = 9x^2 - 30x + 25$$
$$7x^2 - 30x = -27$$
$$x^2 - \frac{30}{7}x = -\frac{27}{7}$$
$$x^2 - \frac{30}{7}x + \left(\frac{15}{7}\right)^2 = \left(\frac{15}{7}\right)^2 - \frac{27}{7} = \frac{36}{49}$$
$$x - \frac{15}{7} = \sqrt{\frac{36}{49}} = \pm \frac{6}{7},$$
$$\therefore x = \pm \frac{6}{7} + \frac{15}{7} = 3 \text{ or } \frac{9}{7}.$$

8. If $2x^2 - 5x = 12$, what is the value of x?

$$x^2 - \frac{5}{2}x = 6$$
$$x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = 6 + \frac{25}{16} = \frac{121}{16}$$
$$x - \frac{5}{4} = \sqrt{\frac{121}{16}} = \pm \frac{11}{4},$$
$$\therefore x = \frac{5}{4} \pm \frac{11}{4} = 4 \text{ or } -\frac{3}{2}.$$

9. If $3x^2 - 8x = 5$, what is the value of x?

$$x^2 - \frac{8}{3}x = \frac{5}{3}$$

$$x^2 - \frac{8}{3}x + \left(\frac{4}{3}\right)^2 = \frac{5}{3} + \frac{16}{9} = \frac{31}{9}$$

$$x - \frac{4}{3} = \sqrt{\frac{31}{9}} = \pm\frac{1}{3}\sqrt{31},$$

$$\therefore x = \frac{4}{3} \pm \frac{1}{3}\sqrt{31} = \frac{4}{3} + \frac{1}{3}\sqrt{31} \text{ or } \frac{4}{3} - \frac{1}{3}\sqrt{31},$$

$$\text{or } x = \frac{1}{3}(4 \pm \sqrt{31}) = \frac{1}{3}(4 + \sqrt{31}) \text{ or } \frac{1}{3}(4 - \sqrt{31}).$$

Or actually extracting the square root of 31, we find $\sqrt{31} = 5{\cdot}568$ nearly, and hence

$$\therefore x = \frac{1}{3}(4 \pm 5{\cdot}568) = \frac{9{\cdot}568}{3} \text{ or } -\frac{1{\cdot}568}{3} = 3{\cdot}189 \text{ or } -{\cdot}522.$$

10. If the proportion $x^2 : 3x - 2 = 7x - 4 : 15$, what is the value of x?

By art. (244); $21x^2 - 26x + 8 = 15x^2$

$$6x^2 - 26x = -8$$

$$x^2 - \frac{13}{3}x = -\frac{4}{3}$$

$$x^2 - \frac{13}{3}x + \left(\frac{13}{6}\right)^2 = \frac{169}{36} - \frac{4}{3} = \frac{121}{36}$$

$$x - \frac{13}{6} = \sqrt{\frac{121}{36}} = \pm\frac{11}{6},$$

$$\therefore x = \frac{13}{6} \pm \frac{11}{6} = 4 \text{ or } \frac{1}{3}.$$

EXERCISES.

1. If $x^2 - 6x = 7$,	. . . .	$x = 7$ or -1
2. ... $x^2 + 8x = 9$,	. . . .	$x = 1$ or $-$
3. ... $x^2 + 7x = 44$,	. . . .	$x = 4$ or -1
4. ... $x^2 - 7x = 44$,	. . . .	$x = 11$ or $-$
5. ... $x^2 + 15 = 8x$,	. . . .	$x = 5$ or
6. ... $x^2 + 5x + 6 = 2$,	. .	$x = -1$ or -4
7. ... $x^2 - 5x + 6 = 2$,	. . .	$x = 1$ or 4
8. ... $x^2 - 13x + 4 = 18$,	. .	$x = 14$ or

9. If $x^2 + 32x = 320$, ... $x = 8$ or -40.

10. ... $x^2 - 7x = -12$, . . . $x = 3$ or 4.

11. ... $x^2 - 13x = 68$, ... $x = 17$ or -4.

12. ... $x^2 + 7x = 8$, ... $x = 1$ or -8.

13. ... $2x^2 + 3x = 65$, ... $x = 5$ or $-6\frac{1}{2}$.

14. ... $\frac{x^2}{100} = x - 24$, ... $x = 60$ or 40.

15. ... $x^2 - x - 40 = 170$, . . . $x = 15$ or -14.

16. ... $2x^2 - 16x + 20 = 38$, . . . $x = 9$ or -1.

17. ... $3x^2 + 2x = 85$, ... $x = 5$ or $-5\frac{2}{3}$.

18. ... $\frac{2x^2}{3} - \frac{5x}{2} = \frac{2}{3}$, ... $x = 4$ or $-\frac{1}{4}$.

19. ... $\frac{x}{4} - \frac{44}{x-2} = 4$, ... $x = 24$ or -6.

20. ... $\frac{5}{6}x^2 - \frac{1}{2}x + \frac{3}{4} = 8 - \frac{2}{3}x - x^2 + \frac{273}{12}$, $x = 4$ or $-4\frac{1}{11}$.

21. ... $6x^2 - 37x + 57 = 0$, . . . $x = 3$ or $3\frac{1}{6}$.

22. ... $x^2 - 7x + 10 = 0$, . . . $x = 2$ or 5.

279. If m be taken for $\frac{b}{a}$, and n for $\frac{c}{a}$, the equations in the four cases in art. (274), after dividing them by a, become

$$x^2 + mx = n \quad \ldots \quad (1),$$
$$x^2 - mx = n \quad \ldots \quad (2),$$
$$x^2 + mx = -n \quad \ldots \quad (3),$$
$$x^2 - mx = -n \quad \ldots \quad (4).$$

Taking the first of these, and completing the square,

$$x^2 + mx + \left(\frac{m}{2}\right)^2 = \frac{m^2}{4} + n = \frac{m^2 + 4n}{4}.$$

Extracting the square root,

$$x + \frac{m}{2} = \sqrt{\frac{m^2 + 4n}{4}} = \frac{1}{2}\sqrt{(m^2 + 4n)},$$

$$x = -\frac{m}{2} \pm \frac{1}{2}\sqrt{(m^2 + 4n)} \quad \ldots \quad [1].$$

Performing the same process for the second case, it would be found that

$$x = \frac{m}{2} \pm \frac{1}{2}\sqrt{(m^2 + 4n)} \quad \ldots \quad [2],$$

for the third,

$$x = -\frac{m}{2} \pm \frac{1}{2}\sqrt{(m^2 - 4n)} \quad \ldots \quad [3],$$

and for the fourth,

$$x = \frac{m}{2} \pm \frac{1}{2}\sqrt{(m^2 - 4n)} \quad \ldots \quad [4].$$

These four results are comprised in the expression

$$x = \pm\frac{m}{2} \pm \frac{1}{2}\sqrt{(m^2 \pm 4n)},$$

taking the signs in any order.

280. In each of these four cases the unknown quantity has two values, one of these being obtained by taking the radical part positive, and the other by taking it negative. In the first and second cases, one of the roots will be positive, and the other negative, for the radical part being greater than $\sqrt{m^2}$, or m, its value multiplied by $\frac{1}{2}$ will exceed $\frac{m}{2}$. In the third and fourth cases, the radical part is less than m, and its half, therefore, less than $\frac{m}{2}$; and hence in [3] the two values of x are negative, and in [4] they are positive. When, however, m^2 is less than $4n$, the quantity under the radical sign in the third and fourth cases becomes negative, and its square root impossible, or the two roots are *imaginary;* in other circumstances, they are *real.*

281. By means of these four formulas for the value of x, its numerical value in any particular case may be easily found, by substituting for m and n their values in any given equation, after it is reduced to the form

$$x^2 + mx = n.$$

Thus, taking the second of the preceding examples,

$$x^2 + 6x = 40.$$

Here $m = 6$, $n = 40$, and hence by [1],

$$x = -\frac{6}{2} \pm \frac{1}{2}\sqrt{(6^2 + 4 \times 40)}$$

$$= -3 \pm \frac{1}{2}\sqrt{(36 + 160)} = -3 \pm \frac{1}{2}\sqrt{196}$$

$$= -3 \pm \frac{14}{2} = -3 \pm 7 = 4 \text{ or } -10.$$

Taking the first of these examples,

$$x^2 - 4x = 12.$$

Here $m = -4$, $n = 12$, therefore the value of x in the formula [2] may be taken, which gives

$$x = \frac{4}{2} \pm \frac{1}{2}\sqrt{(16 + 48)} = 2 \pm \frac{1}{2}\sqrt{64} = 6 \text{ or } -2,$$

for the sign of m being given to the second term in (2), m is to be considered as positive in [2].

If the two roots in [1] of equation (1) in (279) be added, and also multiplied together, it will be found that their sum, with its sign changed, is the coefficient (m) of the second term, and the product is the absolute term ($-n$).

The equation is $\quad x^2 + mx - n = 0.$

The roots are $\quad -\frac{m}{2} + \frac{1}{2}\sqrt{(m^2 + 4n)},$

and $\quad -\frac{m}{2} - \frac{1}{2}\sqrt{(m^2 + 4n)}.$

The sum of the roots is evidently $= -m$, and their product (69, THEO. III.)

$$= \frac{m^2}{4} - \frac{1}{4}(m^2 + 4n) = \frac{m^2}{4} - \frac{m^2}{4} - n = -n.$$

When $m^2 = 4n$, the radical term in [3] and [4] disappears. For example, let

$$x^2 \pm 8x = -16,$$

then $m = 8$, $n = 16$, and

$$x = \mp \frac{8}{2} \pm \frac{1}{2}\sqrt{(64 - 4 \times 16)} = \mp 4 \pm \frac{1}{2}\sqrt{0} = \mp 4,$$

where -4 is the root when 8 is positive, that is, for case [3]; and $+4$ is the root when 8 is negative, that is, for case [4].

282. In this case—that is, when the radical term in the value of the roots vanishes—the two roots are equal.

When $m^2 \angle 4n$, the roots of [3] and [4] are imaginary. As an example, let

$$x^2 \pm 10x = -30,$$

then $m = \pm 10$, $n = 30$, and

$$x = \mp \frac{10}{2} \pm \frac{1}{2}\sqrt{(100 - 4 \times 30)} = \mp 5 \pm \frac{1}{2}\sqrt{-20}$$

$$= \mp 5 \pm \frac{1}{2}\sqrt{4}\sqrt{-5} = \mp 5 \pm \frac{2}{2}\sqrt{-5} = \mp 5 \pm \sqrt{-5},$$

or since $\sqrt{-5} = \sqrt{5}\sqrt{-1}$, $x = \mp 5 \pm \sqrt{5}\sqrt{-1}$.

283. These imaginary values always indicate something impossible in the question, the conditions of which are expressed by the equation.

To illustrate further these four formulas, the reader may perform by them the preceding exercises.

SECOND METHOD.

284. Rule I. Having reduced the equation to the usual form, multiply all its terms by four times the coefficient of the first term of the first member; and then add to both sides of the result the square of the coefficient of the second term; the first member will then be a complete square. Extract the square root of both sides, and the result will be a simple equation.

EXAMPLE.

Let $$2x^2 - 5x = 12,$$

multiply by (4×2), or 8,

$$16x^2 - 40x = 96;$$

add 5^2 or 25, $$16x^2 - 40x + 5^2 = 96 + 25 = 121,$$

$$4x - 5 = \sqrt{121} = \pm 11,$$

$$4x = 5 \pm 11 = 16 \text{ or } -6,$$

$$x = 4 \text{ or } -\frac{3}{2}.$$

The first two operations may be easily performed at the same time; thus,

$$2x^2 - 5x = 12;$$

multiply by 8, and add 25,

$$16x^2 - 40x + 25 = 96 + 25 = 121,$$

$$4x - 5 = \sqrt{121} = \pm 11,$$

$$4x = 5 \pm 11 = 16 \text{ or } -6,$$

$$x = 4 \text{ or } -\frac{3}{2}.$$

One advantage of this method is, that no fractions are introduced into the solution, as often happens by the first method.

The principle of the rule is evident in solving the equation

$$ax^2 + bx = c \qquad \dots \quad (1)$$

by the first method. For the process of solution gives

$$x^2 + \frac{b}{a}x = \frac{c}{a},$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{c}{a} = \frac{b^2 + 4ac}{4a^2},$$

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 + 4ac}{4a^2}} = \frac{\sqrt{(b^2 + 4ac)}}{2a}.$$

Now multiply both sides by $2a$, and the result is

$$2ax + b = \sqrt{(b^2 + 4\,ac)} \qquad \dots \quad (2);$$

squaring,

$$4a^2x^2 + 4abx + b^2 = b^2 + 4ac \quad \dots \quad (3);$$

and this latter equation is found from the given one $ax^2 + bx = c$, by performing the first two processes of the second method;

thus, $\qquad ax^2 + bx = c;$

multiplying by $4a$,

$$4a^2x^2 + 4abx = 4ac;$$

adding b^2,

$$4a^2x^2 + 4abx + b^2 = b^2 + 4ac \quad \dots \quad (4);$$

and this is the equation (3), which is the square of (2). The square root of (4) is therefore (2), or

$$2ax + b = \sqrt{(b^2 + 4ac)},$$

from which the value of x is found to be

$$x = \frac{1}{2a}\{-b \pm \sqrt{(b^2 + 4ac)}\}.$$

The exercises under the first method may be performed by this also.

EXAMPLES WITH LITERAL COEFFICIENTS.

1. If $ax^2 - dx - e = bx^2 - cx - f$, what is the value of x?

$$(a - b)x^2 + (c - d)x = e - f.$$

By first method, $\qquad x^2 + \frac{c - d}{a - b}x = \frac{e - f}{a - b},$

$$x^2 + \frac{c - d}{a - b}x + \frac{(c - d)^2}{4(a - b)^2} = \frac{e - f}{a - b} + \frac{(c - d)^2}{4(a - b)^2}$$

$$= \frac{(c - d)^2 + 4(a - b)(e - f)}{4(a - b)^2};$$

hence $$x + \frac{c-d}{2(a-b)} = \sqrt{\frac{(c-d)^2 + 4(a-b)(e-f)}{4(a-b)^2}}$$

$$= \frac{\sqrt{\{(c-d)^2 + 4(a-b)(e-f)\}}}{2(a-b)}$$

$$x = \frac{1}{2(a-b)}\{d - c \pm \sqrt{[(c-d)^2 + 4(a-b)(e-f)]}\},$$

which are the two values of x; and its numerical values for any particular values of these coefficients would be found by substituting for them their values.

Solving the equation by the second method,

multiply by $4(a-b)$,

$$4(a-b)^2x^2 + 4(a-b)(c-d)x = 4(a-b)(e-f);$$

adding $(c-d)^2$,

$$4(a-b)^2x^2 + 4(a-b)(c-d)x + (c-d)^2$$
$$= (c-d)^2 + 4(a-b)(e-f);$$

extracting the square root,

$$2(a-b)x + c - d = \sqrt{\{(c-d)^2 + 4(a-b)(e-f)\}},$$

or $$x = \frac{1}{2(a-b)}\{d - c \pm \sqrt{[(c-d)^2 + 4(a-b)(e-f)]}\}.$$

EXERCISES.

1. If $x^2 + x = a$, . $x = \frac{1}{2}\{\pm\sqrt{(1+4a)} - 1\}$.

2. ... $ax^2 - bx = d - c$, . $x = \frac{1}{2a}\{b \pm \sqrt{b^2 + 4a(d-c)}\}$.

3. ... $4a^2 - 2x^2 + 2ax = 18ab - 18b^2$, $x = 2a - 3b$, or $-a + 3b$.

4. ... $\frac{m^2x^2}{n^2} - x^2 = a^2 + b^2 - 2bx$,

$$x = \frac{n}{m^2 - n^2}\{\pm\sqrt{(a^2m^2 - a^2n^2 + b^2m^2)} - bn\}.$$

5. ... $adx - acx^2 = bcx - bd$, ... $x = \frac{d}{c}$, or $-\frac{b}{a}$.

6. ... $mqx^2 - mnx + pqx = np$, . . . $x = \frac{n}{q}$, or $-\frac{p}{m}$.

7. ... $\frac{a - \sqrt{2ax - x^2}}{a + \sqrt{2ax - x^2}} = \frac{x}{a - x}$, . . $x = a$, or $\frac{a}{5}$.

In the third example, the radicals disappear, as the quantity under the radical sign is a complete square.

285. Rule II. Equations in which the exponent of one of the powers of the unknown quantity is double that of the other, may be solved by means of the same method as quadratics.

For if a new unknown quantity be assumed equal to the lower power of that in the given equation, the square of this quantity will be equal to the higher power, and the transformed equation is therefore a quadratic. Whether the unknown quantity be simple or compound, this rule is applicable.

EXAMPLES.

1. If $x^6 - 2x^3 = 48$, what is the value of x?

Let $z = x^3$, then $x^6 = z^2$, and the equation becomes

$$z^2 - 2z = 48.$$

Completing the square,

$$z^2 - 2z + 1 = 48 + 1 = 49,$$
$$z - 1 = \sqrt{49} = \pm 7,$$
$$z = 1 \pm 7 = 8 \text{ or } -6.$$

Having found the value of z, that of x is easily obtained. For

$$x^3 = z, \therefore x = \sqrt[3]{z};$$

therefore

$$x = \sqrt[3]{8} = 2 \text{ or } x = \sqrt[3]{-6} = \sqrt[3]{6}\sqrt[3]{-1} = -1\sqrt[3]{6} = -\sqrt[3]{6};$$

and hence the two values of x are

$$x = 2, \text{ and } x = -\sqrt[3]{6}.$$

286. It is shewn in the theory of equations, that every equation has as many roots as there are units in its degree; and hence x has three values in the equation $x^3 = z$, and will have three values for every value assigned to z. Therefore since z has two values here, x will have six, as it ought, since the given equation is of the sixth degree.

The equation may be solved without using a new quantity, as z; thus—

completing the square,

$$x^6 - 2x^3 + 1 = 48 + 1 = 49;$$

extracting the square root,

$$x^3 - 1 = \sqrt{49} = \pm 7,$$
$$x^3 = 1 \pm 7 = 8 \text{ or } -6;$$

hence $\qquad x^3 = 8$ or $x = \sqrt[3]{8} = 2$,

and $x^3 = -6$ or $x = \sqrt[3]{-6} = \sqrt[3]{-1}\sqrt[3]{6} = -1\sqrt[3]{6} = -\sqrt[3]{6}$.

When the unknown quantity is compound, as in the next example, the expressions are much simplified by using a new unknown quantity.

2. Given $\sqrt{(x^2 - 3x - 1)} + 6 = x^2 - 3x - 1$.

Let $z = \sqrt{(x^2 - 3x - 1)}$, then $x^2 - 3x - 1 = z^2$,

and
$$z^2 - z = 6,$$
$$z^2 - z + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4},$$
$$z - \frac{1}{2} = \sqrt{\frac{25}{4}} = \frac{5}{2},$$
$$\therefore\ z = \frac{1}{2} \pm \frac{5}{2} = 3 \text{ or } -2.$$

In order to find the value of x, one of the values of z^2 must now be equated with $x^2 - 3x - 1$; thus, taking the first value of z or 3,
$$x^2 - 3x - 1 = z^2 = 9;$$
and from this equation, by the usual process, is found
$$x = 5 \text{ or } -2.$$

But by taking the other value of z, namely, -2, the equation is
$$x^2 - 3x - 1 = z^2 = (-2)^2 = 4.$$
From which is found $x = \frac{3}{2} \pm \frac{1}{2}\sqrt{29}$.

Hence x has four values; namely, 5, -2, $\frac{3}{2} + \frac{1}{2}\sqrt{29}$, and $\frac{3}{2} - \frac{1}{2}\sqrt{29}$; and on trial it will be found that they all satisfy the given equation; and, in fact, if the given equation be cleared of its radical terms by squaring, it will be found to be of the fourth degree, and therefore it has four roots.

When the unknown quantity is compound, and is under the sign of the square root, and only the terms of it, which contain the simple unknown quantity, are equal to the corresponding integral terms, the compound integral quantity may be made equal to that under the radical sign, by adding and subtracting a proper quantity, so that the equality may still exist. Thus,

3. Let $\sqrt{(2x^2 - 3x + 5)} = 2x^2 - 3x - 15$;

adding to the second number 20, and also taking 20 from it; that is, adding $20 - 20$,
$$\sqrt{(2x^2 - 3x + 5)} = 2x^2 - 3x + 5 - 20.$$
Now, let $z^2 = 2x^2 - 3x + 5$, and the equation becomes
$$z = z^2 - 20,$$
or
$$z^2 - z = 20;$$
and by the usual process of solution it will be found that
$$z = 5 \text{ or } -4.$$

Hence for $z = 5$, $2x^2 - 3x + 5 = 25$;

and from this equation is found $x = 4$ or $-\frac{5}{2}$.

Also for $z = -4$, $2x^2 - 3x + 5 = 16$;

and from this is found $x = \frac{3}{4} \pm \frac{1}{4}\sqrt{97}$,

so that the four values of x are 4, $-\frac{5}{2}$, $\frac{3}{4} + \frac{1}{4}\sqrt{97}$, and $\frac{3}{4} - \frac{1}{4}\sqrt{97}$.

4. If $x^{2n} - 2ax^n = b^2$, what is the value of x?

Assume $x^n = z$, then $x^{2n} = z^2$, and the equation becomes

$$z^2 - 2az = b^2;$$

and from this equation is easily found

$$z = a \pm \sqrt{(a^2 + b^2)};$$

hence $\qquad x = \sqrt[n]{z} = \sqrt[n]{\{a \pm \sqrt{(a^2 + b^2)}\}} \quad \ldots \quad (1).$

As a particular example, let $n = 2$, $a = 4$, $b = 3$, and the given equation becomes

$$x^4 - 8x^2 = 9,$$

from which the values of x may be found in the usual way, by assuming $x^2 = z$; but these values are more readily found from the preceding general formula (1), which becomes

$$x = \sqrt{\{4 \pm \sqrt{(16 + 9)}\}} = \sqrt{(4 \pm 5)} = \sqrt{9} \text{ or } \sqrt{-1}$$

$$= \pm 3 \text{ or } \pm \sqrt{-1}.$$

EXERCISES.

1. If $x^4 - 6x^2 + 10 = 2$, . . $x = \pm 2$ or $\pm \sqrt{2}$.

2. ... $x^6 - 4x^3 - 28 = 4$, . . $x = 2$ or $-\sqrt[3]{4}$.

3. ... $x^{4n} - 2x^{2n} = a$, . . . $x = \{1 \pm (1 + a)^{\frac{1}{2}}\}^{\frac{1}{2n}}$.

4. ... $5x^2 - 2x = \sqrt{(5x^2 - 2x)} + 12$,

$x = 2$ or $-\frac{8}{5}$ or $\frac{1}{5}(1 \pm \sqrt{46})$.

5. ... $4\sqrt{(x^2 - 3x)} = x^2 - 3x$, . $x = 3$ or 0 or $\frac{1}{2}(3 \pm \sqrt{73})$.

6. ... $2\sqrt{(x^2 - 3x + 11)} = x^2 - 3x + 8$,

$x = 2$ or 1 or $\frac{1}{2}(3 \pm \sqrt{-31})$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

EXAMPLES.

1. Find a number such that, if 8 times the number be added to its square, the sum shall be $= 65$.

Let x = the number,

then $$x^2 + 8x = 65;$$

and the values of x are found by the usual process of solution to be

$$x = -4 \pm 9 = 5 \text{ or } -13.$$

Either of these values satisfies the equation. If, however, the negative solution -13 be considered as the positive number 13 to be subtracted, then the enunciation of the question would require to be so far modified as was done in simple equations, by changing the condition of addition into subtraction (252); the cause of which change is evident, by substituting $-x$ for x in the equation. But the enunciation without any change will be adapted to both values, by observing that the algebraic addition of a negative quantity is the same as the arithmetical subtraction of a positive one (56).

2. Find a number such that, if 4 times its square be diminished by 6 times the number itself, the remainder shall be $= 70$.

Let x = the number,

then $$4x^2 - 6x = 70;$$

and from this equation is found

$$x = 5 \text{ or } -\frac{7}{2}.$$

3. Divide the number 24 into two parts, such that their product shall be $= 95$.

Let x = one of the parts, then $24 - x$ = the other,

and $$x(24 - x) = 95,$$

from which is found $$x = 19 \text{ or } 5.$$

4. Find a number such that, if 15 be added to its square, the sum shall be $= 8$ times the number.

Let x = the number,

then $$x^2 + 15 = 8x,$$

from which is found $$x = 5 \text{ or } 3.$$

287. It appears from this example, that a question producing a quadratic equation sometimes admits of two positive solutions,

which belong literally to the enunciation without any modification. Positive solutions are also called *direct*, and negative ones, *indirect.*

5. To divide a number (a) into two parts, such that their product shall be $= b^2$.

Let $x =$ one of the numbers,

then $\quad a - x =$ the other,

and $\quad x(a - x) = b^2.$

$$\therefore\ x = \frac{1}{2}\{a \pm \sqrt{(a^2 - 4b^2)}\}.$$

This expression for the value of x shews the limitation of the conditions; for its value is possible, provided that $4b^2$ does not exceed a^2. When $4b^2 = a^2$, or $b = \frac{a}{2}$, the value of $x = \frac{a}{2}$, and that of $a - x = a - \frac{a}{2} = \frac{a}{2}$; so that in this case the two parts are equal, each being a half of the given number. This is also the greatest value that b can have; and hence the product is greatest when the number is divided into two equal parts.

This result is analogous to the 27th proposition of the 6th book of Euclid.

6. There are three numbers in continued proportion; the sum of the first and second is $= 10$, and the third exceeds the second by 24: what are the numbers?

Let $x =$ the least,

then $\quad 10 - x = \ldots$ second,

and $\quad 34 - x = \ldots$ third;

also (322) $\quad x : 10 - x = 10 - x : 34 - x,$

whence $\quad x^2 - 20x + 100 = 34x - x^2,$

$$\therefore\ x = \frac{27}{2} \pm \frac{23}{2} = 25 \text{ or } 2.$$

When $x = 2$, $10 - x = 8$, $34 - x = 32$, and the numbers are therefore 2, 8, and 32.

When $x = 25$, $10 - x = -15$, $34 - x = 9$, and the numbers are then 25, -15, and 9.

7. A person bought a number of oxen for £80, and if he had bought 4 more for the same money, the price of each would have been £1 less: how many did he purchase?

Let $x =$ the number of oxen,

then $\frac{80}{x} =$ the price paid for each,

and $\frac{80}{x+4} =$ the price that would have been paid for each, had more been bought for the same money.

Hence by the question, $\frac{80}{x} = \frac{80}{x+4} + 1$;

and from this equation is found by the rule

$$x = \pm 18 - 2 = 16 \text{ or } - 20.$$

The number of oxen bought were therefore 16.

The negative value $- 20$ to this question suggests, as usual, th necessity of modifying the conditions of the question, in orde that the value 20 may be literally or directly applicable. Th necessary modification to be made will be perceived by substi tuting the negative value in the preceding equation, when it gives

$$\frac{80}{-x} = \frac{80}{-x+4} + 1,$$

or $$\frac{80}{x-4} - \frac{80}{x} = 1,$$

which shews that the condition of buying 4 *more* must be change into that of buying 4 *fewer*; and hence, also, the price, instead o being £1 less, will now be £ more. The question thus modifie has for its direct solution the value 20, and its enunciation is:—

8. A person bought a number of oxen for £80, and if he ha bought 4 fewer for the same money, the price would have been £ more for each: how many did he purchase?

Let $x =$ the number of oxen,

then $\frac{80}{x} =$ the price of each,

and $\frac{80}{x-4} =$ the price of each had 4 fewer been bought.

Hence $$\frac{80}{x-4} - \frac{80}{x} = 1,$$

whence $$x = \pm 18 + 2 = 20 \text{ or } - 16.$$

The positive solution 20 is the direct solution of this questi and the second value, when taken positively, is the solution of th question properly modified—that is, of the seventh example.

Thus, the two values of the unknown quantity are the solution of two analogous questions, which are mutually convertible by

proper modification of those conditions which depend on addition and subtraction.

As the transactions of buying and selling stand in a relation to each other similar to that of addition and subtraction, the former question is susceptible of another modification which adapts it to the second solution 20. The enunciation of it is this:—

9. A person sold a number of oxen for £80, and if he had sold 4 fewer for the same money, the price would have been £1 more for each: how many did he sell?

The reader will find by solving this question that the values of x are 20 and -16; the former of which is the solution of this question; and the latter, or 16, is the solution of a modification of this question, which is the seventh example. The mere change of buying into selling, or the converse, does not, however, alter the solution, unless the corresponding changes be made on the other conditions. This question shews that the *indirect* solution -20 in example seventh, are 20 sold; that is, the negative solution is to be taken in a sense directly opposite to that of the positive solution.

10. Several persons are bound to pay the expense of a law process, which amounts to £800; but three of them being insolvent, the rest have £60 each to pay additional: how many persons were concerned?

Let $x =$ the number of persons,

then $\quad \dfrac{800}{x} =$ the share each had originally to pay,

and $\quad \dfrac{800}{x-3} =$ ... ... actually ... ;

$$\therefore \frac{800}{x-3} = \frac{800}{x} + 60,$$

whence $\quad x = \dfrac{3}{2} \pm \dfrac{13}{2} = 8$ or -5.

From the former illustrations, the reader will easily interpret the negative solution. The -5 means five persons who are to receive equal shares of £800.

11. Given the fraction $\dfrac{a}{b}$ to find a number such that, if it be added to the numerator, and then to the denominator, the former resulting fraction shall be m times the latter.

Let $x =$ the required number,

then $$\frac{a+x}{b} = \frac{ma}{b+x},$$

$$\therefore x = \pm \frac{1}{2}\sqrt{\{4abm + (a-b)^2\}} - \frac{1}{2}(a+b).$$

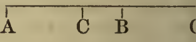
As a particular example, let $a = 4$, $b = 5$, $m = \frac{3}{2}$,

then $$x = \pm \frac{11}{2} - \frac{9}{2} = 1 \text{ or } -10,$$

and $$\frac{4+1}{5} = \frac{3}{2} \; \frac{4}{5+1}, \text{ or } \frac{5}{5} = \frac{3}{2} \cdot \frac{2}{3} = 1.$$

12. Two lights, whose intensities are as 4 to 1, are 3 yarc distant from each other: find that point in the straight lin between them, which is equally enlightened by each.

Let A, B, be the two lights, and C the required point, also



$$\text{let} \quad AC = x,$$

$$\text{then } BC = 3 - x,$$

then, since the intensities of the same light at different distance are inversely as the squares of these distances, and the intensit of the light A is 4 times that of B,

$$\frac{1}{(3-x)^2} = \frac{4}{x^2},$$

This equation produces the quadratic

$$x^2 - 8x + 12 = 0, \text{ or the simple equation } \frac{1}{3-x} = \pm \frac{1}{x}.$$

From either of which $x = 2$ or 6.

The value 2 refers to the point C lying between the lights, an the other value 6 belongs to a point C′ lying beyond B, and a either of these points the intensities of the lights are equal.

In order to generalise the question, so that the solution ma comprehend cases of any distance of the lights, and any intensitie let

$$AB = a, \; AC = x, \text{ then } BC = a - x;$$

also let the intensities of the lights A and B at the distance 1 b denoted respectively by m^2 and n^2.

Then, since the intensities at C are to be equal,

$$\frac{m^2}{x^2} = \frac{n^2}{(a-x)^2} \text{ or } \frac{m}{x} = \pm \frac{n}{a-x},$$

whence $$x = \frac{ma}{m+n} \text{ or } \frac{ma}{m-n}.$$

The former value refers to C, and the latter to C′; for the forme value is less than a, and the latter is greater.

The corresponding values of $a - x$ are $\frac{an}{m + n}$ and $- \frac{an}{m - n}$ or $\frac{an}{n - m}$; these values being BC and BC′, observing that $a - x$ is reckoned from B, and that it is positive when $x \angle a$—that is, when C is between B and A; and hence it is negative when C′ lies beyond B, for which $x \triangleright a$.

EXERCISES.

1. Find a number such that its square minus 6 times the number itself shall be $= 7$, ... $= 7$ or $- 1$.

2. Find a number such that its square plus 8 times the number itself shall be $= 9$, ... $= 1$ or $- 9$.

3. Find a number such that twice its square plus 3 times the number itself shall be $= 65$, ... $= 5$ or $- \frac{13}{2}$.

4. Find a number such that its square minus 1, with $\frac{2}{3}$ of the remainder, shall be $= 5$ times the number divided by 2, $= 2$ or $- \frac{1}{2}$.

5. Find a number such that, if 44 be divided by the number minus 2, the quotient shall be $=$ one-fourth of the number minus 4, ... $= 24$ or $- 6$.

6. Divide the number 49 into two such parts that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater as 16 to 9, ... $= 28$ and 21.

7. Find two numbers whose difference is $= 8$, and their product $= 240$, ... $= 12$ and 20, or $- 20$ and $- 12$.

8. Find two numbers whose sum is $= 100$, and their product $= 2059$, ... $= 29$ and 71.

9. Find two numbers whose sum is $= m$, and their product $= n^2$, ... $= \frac{1}{2}\{m \pm \sqrt{(m^2 - 4n^2)}\}$.

10. The intensities of two lights are as 4 to 9: at what distance from the latter are their intensities equal, supposing their distance to be 12 feet? . . . . $= 7{\cdot}2$ and 36.

11. A certain number of persons equally concerned in a transaction in business lost £960; but four of them becoming insolvent, the rest had to sustain a loss each of £40 greater than he otherwise should have done: how many persons were concerned? . . . . $= 12$ and $- 8$.

12. A store-farmer bought as many sheep as cost him £80, and after reserving 10 for himself, he sold the remainder for £81, and thus gained 2s. a head on them: how many did he purchase? . . . . $= 100$ or $- 80$.

13. What number is that, which, being added, first to the numerator and then to the denominator of the fraction two-fifths, the former result is $= 4$ times the latter? . . $= 3$ or $- 10$

14. A vintner draws a certain quantity of wine out of a full vessel that contains 256 gallons; and then filling the vessel with water draws off the same quantity of liquor as before, and so on for four draughts, after which there were only 81 gallons of pure wine left how much wine did he draw each time? $= 64, 48, 36$, and 27 gallons

15. A horse-dealer pays a certain sum for a horse, which he afterwards sells for £144, and gains exactly as much per cent. as the horse cost him: what did he pay for the horse? $=$ £80

16. In a parcel containing 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins and the whole parcel is worth 18 shillings: how many are there of each kind? . . . $= 6$ of silver and 18 of copper

17. Find a number such that, if 9 be taken from its square, the remainder shall exceed 100 by as much as the number itself is less than 23, ... $= 11$ or $- 12$

18. Find a number such that, if 18 be added to the product of its half by its third, the sum shall be $= 4$ times the number, $= 6$ or 18

19. A horse-dealer bought a number of horses for £180, but had he bought 3 horses more for the same sum, he would have paid £3 less for each: how many horses did he buy? . $= 12$ or $- 15$

20. In attempting to arrange a number of counters in the form of a square, it was found that there were 7 over; and when the side of the square was increased by one, there was a deficiency of 8 to complete the square: required the number of counters, $= 56$

21. Two messengers (A and B) being despatched at the same time to a place 90 miles distant; A rode one mile an hour more than B and arrived at the end of his journey an hour before him: at what rate per hour did each travel? A $= 10$ and B $= 9$ miles per hour

22. A grazier bought a certain number of oxen for £240; after losing 3, he sold the remainder at £8 a head more than they cost him, and thus gained £59 by the bargain: what number did he buy? . . . . $= 1($

QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

288. When there are two equations, and two unknown quantities, they cannot always be solved by the preceding method for their solution will in general depend on an equation of the fourth degree containing one unknown quantity (295). There are however, several species of such equations, the solution of which ultimately depends on that of a quadratic containing only one unknown quantity.

A *system of equations* consists of two or more equations containing unknown quantities that have the same value; and a *system of values* of unknown quantities is a set of values that satisfy a system of equations.

CASE I.—WHEN ONE OF THE EQUATIONS IS SIMPLE.

289. RULE. Find a value of one of the unknown quantities from the simple equation, and substitute this value in the other equation; the result will be a quadratic with one unknown quantity, which may be solved by the preceding rules.

EXAMPLES.

1. If $\left\{ \begin{array}{l} x - y = 2 \\ x^2 + y^2 = 100 \end{array} \right\}$ what are the values of x and y?

By the first equation, $x = y + 2$ ... [1].

Substituting this value of x in the second, we have

$$(y + 2)^2 + y^2 = 100 \quad \ldots \quad [2],$$

whence $y = 6$ or -8;

hence by [1], $x = 8$ or -6.

There are therefore two systems of values of x and y, which are respectively 8 and 6, or -6 and -8, which may be arranged thus—

$$x = 8, -6$$

$$y = 6, -8.$$

Equations may frequently be solved by means of particular analytical artifices. The last example, for instance, may be solved thus—

Squaring the first equation, we find $x^2 - 2xy + y^2 = 4$.

Subtracting this from the second, $2xy = 96$,

or $xy = 48$ ... [3].

Substituting for x in this equation its value in [1],

$$(y + 2)y = 48,$$

or $y^2 + 2y = 48$,

from which the same values of y are obtained as from [2], and hence those of x may be found from [3].

Also, if the square of the first be subtracted from twice the second, the remainder will be $x^2 + 2xy + y^2 = 196$; hence (69, THEO. I.) $(x + y)^2 = 196$ and $x + y = \pm 14$: by combining this with the first equation, the same system of values of x and y will be obtained as before.

2. If $\left\{\begin{array}{l} x + y = 10 \\ x^2 - y^2 = 20 \end{array}\right\}$ what are the values of x and y?

By the first equation, $x = 10 - y \quad \ldots \quad [1].$

Substituting this value of x in the second,

$$(10 - y)^2 - y^2 = 20,$$

or $$20y = 80,$$

whence $$y = 4 \quad \ldots \quad [3],$$

and, by [1], $$x = 6.$$

Only one system of values of x and y are thus obtained; but i $y = 4$ be substituted in the second equation, two values of x ar obtained; namely, $x = \pm 6$. The system $x = -6$, $y = 4$, wi satisfy the second equation only, but not the first; and it will b found that no system except the first, $x = 6$, and $y = 4$, wi satisfy both equations.

These equations admit of another simple mode of solution. B dividing the second by the first, the result is

$$x - y = 2.$$

Adding this to the first equation, $2x = 12$, hence $x = 6$.

Subtracting it from the same, $2y = 8$, $\ldots$ $y = 4$.

3. Find the values of x and y in the equations

$$x - y = 2 \quad \ldots \quad [1],$$

$$x^2 - xy + y^2 = 39 \quad \ldots \quad [2].$$

By [1], $$y = x - 2 \quad \ldots \quad [3];$$

substituting this value in [2],

$$x^2 - x(x - 2) + (x - 2)^2 = 39,$$

from which is found, $x = \pm 6 + 1 = 7$ or -5;

hence by [3], $y = 5$ or -7.

4. Find the values of x and y in the equations

$$\frac{3}{4}x - \frac{2}{3}y = 2 \quad \ldots \quad [1],$$

$$\frac{x^2}{4} - \frac{xy}{3} + \frac{y^2}{4} = 9 \quad \ldots \quad [2].$$

By [1], $$y = \frac{9x - 24}{8} \quad \ldots \quad [3];$$

substituting this value in [2], and reducing the equation to th usual form, it becomes

$$147x^2 - 528x = 5184,$$

rom which may be found $x = 8$ or $-\dfrac{216}{49}$;

nd substituting these values in [3], those of y are

$$y = 6 \text{ or } -\frac{390}{49}.$$

The reason of the rule is evident, for the value of one of the nknown quantities found from the simple equation, will contain nly the simple power of the other, and this value being substiuted in the second equation, cannot produce a term containing a igher power than its square; the resulting equation therefore nust be a quadratic.

EXERCISES.

1. If $\left\{\begin{array}{l} x - y = 3 \\ x^2 + y^2 = 89 \end{array}\right\}$ . . . $\left\{\begin{array}{l} x = 8 \text{ or } -5. \\ y = 5 \text{ or } -8. \end{array}\right.$

2. ... $\left\{\begin{array}{l} x + y = 8 \\ x^2 - y^2 = 16 \end{array}\right\}$ . . $\left\{\begin{array}{l} x = 5. \\ y = 3. \end{array}\right.$

3. ... $\left\{\begin{array}{l} x - 2y = 2 \\ x^2 - 7y^2 = 1 \end{array}\right\}$ . . . $\left\{\begin{array}{l} x = 8 \text{ or } 1\frac{1}{3}. \\ y = 3 \text{ or } -\frac{1}{3}. \end{array}\right.$

4. ... $\left\{\begin{array}{l} 3x + 2y = 22 \\ 5x^2 - 3xy + y^2 = 45 \end{array}\right\}$. $\left\{\begin{array}{l} x = 4 \text{ or } \frac{76}{47}. \\ y = 5 \text{ or } \frac{403}{47}. \end{array}\right.$

5. ... $\left\{\begin{array}{l} 2x + y = 14 \\ x^2 - y^2 = 9 \end{array}\right\}$ . . . $\left\{\begin{array}{l} x = 5 \text{ or } 13\frac{2}{3}. \\ y = 4 \text{ or } -13\frac{1}{3}. \end{array}\right.$

6. ... $\left\{\begin{array}{l} ax + by = 2m \\ xy = n \end{array}\right\}$ . . $\left\{\begin{array}{l} x = \dfrac{m}{a} \pm \dfrac{1}{a}(m^2 - abn)^{\frac{1}{2}}. \\ y = \dfrac{m}{b} \mp \dfrac{1}{b}(m^2 - abn)^{\frac{1}{2}}. \end{array}\right.$

7. ... $\left\{\begin{array}{l} x^2y^2 + 4xy = 96 \\ x + y = 6 \end{array}\right\}$ . . $\left\{\begin{array}{l} x = 4, 2, 3 \pm \sqrt{21}. \\ y = 2, 4, 3 \mp \sqrt{21}. \end{array}\right.$

8. ... $\left\{\begin{array}{l} a(x^2 + y^2) - b(x^2 - y^2) = 2a \\ (a^2 - b^2)(x^2 - y^2) = 4ab \end{array}\right\}$ $\left\{\begin{array}{l} x = \pm\left(\dfrac{a+b}{a-b}\right)^{\frac{1}{2}}. \\ y = \pm\left(\dfrac{a-b}{a+b}\right)^{\frac{1}{2}}. \end{array}\right.$

290. An equation is said to be *homogeneous* when its terms contain the same dimensions of the unknown quantities.

CASE II. — WHEN THE EQUATIONS ARE HOMOGENEOUS, EXCEPT THE CONSTANT TERM.

291. RULE. Assume one of the unknown quantities equal to the product of the other by a new or auxiliary unknown quantity; substitute this product instead of it in the two equations; and the equations will now contain this auxiliary quantity and the square of the other unknown one. Then, in each of the new equations, find the value of the square of this unknown quantity, and equate these values; the resulting equation will be a quadratic containing the auxiliary quantity. Find the value of this auxiliary quantity by the ordinary rule: the values of the other two are easily found.

EXAMPLES.

1. Find the values of x and y in the equations

$$x^2 + xy = 10 \quad \ldots \quad [1],$$

$$2xy - y^2 = 3 \quad \ldots \quad [2].$$

Assume $y = vx$, in which v, the auxiliary quantity, is unknown; then substituting this value instead of y in the given equations, they become

$$x^2 + vx^2 = 10 \quad \ldots \quad [3],$$

$$2vx^2 - v^2x^2 = 3 \quad \ldots \quad [4].$$

But by [3], $$x^2 = \frac{10}{1+v} \quad \ldots \quad [5],$$

and by [4], $$x^2 = \frac{3}{2v - v^2};$$

hence equating these values of x^2,

$$\frac{3}{2v - v^2} = \frac{10}{1+v},$$

or $$3 + 3v = 20v - 10v^2.$$

Hence, by the rule (284),

$$v = \pm\frac{13}{20} + \frac{17}{20} = \frac{3}{2} \text{ or } \frac{1}{5};$$

hence also by [5], $x^2 = \dfrac{10}{1+\frac{3}{2}} \text{ or } \dfrac{10}{1+\frac{1}{5}} = 4 \text{ or } \dfrac{25}{3},$

and $\therefore x = \sqrt{4} \text{ or } \sqrt{\frac{25 \times 3}{9}} = \pm 2 \text{ or } \pm \frac{5}{3}\sqrt{3};$

and likewise

$$\therefore y = vx = \frac{3}{2} \times (\pm 2) \text{ or } \frac{1}{5} \times (\pm \frac{5}{3}\sqrt{3}) = \pm 3 \text{ or } \pm \frac{1}{3}\sqrt{3}.$$

Since the first value of v or $\frac{3}{2}$ gave the two values of $x = \pm 2$, these values of v and x must be combined in finding those of y; namely, ± 3. The second value of v or $\frac{1}{5}$, from which the other values of $x = \pm \frac{5}{3}\sqrt{3}$ were found, must also be combined to get the other two values of y or $\pm \frac{1}{3}\sqrt{3}$.

When the equations are both quadratic, the unknown quantities have four systems of values (295). In this case the systems arranged under each other are these —

$$x = 2, -2, \frac{5}{3}\sqrt{3}, \text{ and } -\frac{5}{3}\sqrt{3},$$

$$y = 3, -3, \frac{1}{3}\sqrt{3}, \text{ and } -\frac{1}{3}\sqrt{3},$$

or $$x = \pm 2 \text{ or } \pm \frac{5}{3}\sqrt{3},$$

and $$y = \pm 3 \text{ or } \pm \frac{1}{3}\sqrt{3}.$$

The reason of the rule is evident from the system of quadratics

$$ax^2 + by^2 + cxy = d \quad \ldots \quad [1],$$
$$mx^2 + ny^2 + pxy = q \quad \ldots \quad [2].$$

Substituting in these equations $y = vx$, they become

$$ax^2 + bv^2x^2 + cvx^2 = d,$$
$$mx^2 + nv^2x^2 + pvx^2 = q,$$

which will each afford a value of x^2 in terms of v and v^2, and these values being equated, the resulting quadratic will give two values of v, which being substituted in either of the values of x^2, will give two values of it, and hence four values for x. Then the systems of values of v and x being substituted in $y = vx$, will give four values of y; so that x and y will thus have four systems of values.

EXERCISES.

1. If $\left\{\begin{array}{l} x^2 - \quad y^2 = \ 5 \\ 4x^2 - 3xy \ = 18 \end{array}\right\}$ . . . $\left\{\begin{array}{l} x = \pm\, 3 \text{ or } \pm\, \frac{6}{7}\sqrt{7}. \\ y = \pm\, 2 \text{ or } \pm\, \frac{1}{7}\sqrt{7}. \end{array}\right.$

2. ... $\left\{\begin{array}{l} 4x^2 - \ xy \ + 2y^2 = 10 \\ 2x^2 + 3xy \ - 2y^2 = \ 0 \end{array}\right\}$. $\left\{\begin{array}{l} x = \pm\, 1 \text{ or } \pm\, \sqrt{2}. \\ y = \pm\, 2 \text{ or } \mp\, \frac{1}{2}\sqrt{2}. \end{array}\right.$

In this example, the value of x^2 cannot be found from the second equation after substituting in it $y = vx$; but after dividing it by x^2, the resulting equation will be one from which the value of v may be found.

3. If $\left\{\begin{array}{l} x^2 - 2xy \ - \ y^2 = \ 2 \\ xy + 2y^2 \ = 5 \end{array}\right\}$. $\left\{\begin{array}{l} x = \pm\, 3 \text{ or } \pm\, \frac{3}{7}\sqrt{7}. \\ y = \pm\, 1 \text{ or } \mp\, \frac{5}{7}\sqrt{7}. \end{array}\right.$

4. ... $\left\{\begin{array}{l} x^2 - 2y^2 \ = \ 8 \\ 3y^2 - \ xy \ = \ 4 \end{array}\right\}$ . . $\left\{\begin{array}{l} x = \pm\, 4 \text{ or } \pm\, \frac{8}{7}\sqrt{7}. \\ y = \pm\, 2 \text{ or } \mp\, \frac{2}{7}\sqrt{7}. \end{array}\right.$

292. Quantities are said to be *similarly involved* in an expression, in which they may be interchanged without altering the expression. Such expressions are said to be *symmetrical.*

CASE III.—WHEN THE EQUATIONS ARE SYMMETRICAL.

293. Rule. For the two unknown quantities substitute the sum and difference of two auxiliary unknown quantities, and the resulting equations will contain the additive quantity with its square, and only the square of the subtractive one. Eliminate the square of this quantity from the equations (262), and there will result a quadratic containing the additive quantity, whose value can then be found. Hence the values of the other auxiliary quantity and the original unknown quantities can easily be found.

EXAMPLE.

Find the values of x and y in the equations

$$x^2 + \ y^2 = 13 \qquad \ldots \qquad [1],$$

$$2x - xy + 2y = 4 \qquad \ldots \qquad [2].$$

ubstituting in these equations $x = u + v$, and $y = u - v$, they ecome

$$(u+v)^2 + (u-v)^2 = 13,$$

nd $$2(u+v) - (u+v)(u-v) + 2(u-v) = 4,$$

hich are reduced to

$$2u^2 + 2v^2 = 13 \quad \ldots \quad [3],$$

nd $$4u - u^2 + v^2 = 4 \quad \ldots \quad [4];$$

wice [4], gives $8u - 2u^2 + 2v^2 \quad \ldots \quad [5];$

ence [3] − [5], $4u^2 - 8u = 5;$

nd from this equation is found $u = \pm\frac{3}{2} + 1 = \frac{5}{2}$ or $-\frac{1}{2}$;

ence, by [5], $$v^2 = \frac{1}{4} \text{ or } \frac{25}{4},$$

$$\therefore v = \pm\frac{1}{2} \text{ or } \pm\frac{5}{2};$$

nd combining the values of u with the corresponding values of v, which are derived from them, we find that

$$x = u + v = \frac{5}{2} \pm \frac{1}{2} \text{ or } -\frac{1}{2} \pm \frac{5}{2} = 3, 2, 2 \text{ or } -3,$$

nd $$y = u - v = \frac{5}{2} \mp \frac{1}{2} \text{ or } -\frac{1}{2} \mp \frac{5}{2} = 2, 3, -3 \text{ or } 2,$$

which are the four systems of values of x and y.

It is evident, from the manner in which these values of x and y are obtained from those of u and v, that the first value of x must be equal to the second of y, and the second of x to the first of y. Likewise, that when u and v have each two values, x and y will have four, and the third value of x will be equal to the fourth of y, and the fourth of x to the third of y; so that when the values of one of them are known, those of the other are also known.

EXERCISES.

1. If $\left\{ \begin{array}{l} x^2 + \ \ y^2 = 25 \\ xy = 12 \end{array} \right\} \qquad \left\{ \begin{array}{l} x = \ \ 4, 3, -3, -4. \\ y = \ \ 3, 4, -4, -3. \end{array} \right.$

2. ... $\left\{ \begin{array}{l} x - \ xy \ + y \ = 1 \\ 2x^2 + 2y^2 \ = 20 \end{array} \right\} \qquad \left\{ \begin{array}{l} x = 3, 1, 1, -3. \\ y = \ \ 1, 3, -3, 1. \end{array} \right.$

3. ... $\left\{ \begin{array}{l} x^2 - 3xy \ + y^2 = -1 \\ 2x^2 + 2y^2 \ = 10 \end{array} \right\} \qquad \left\{ \begin{array}{l} x = \ \ 2, 1, -1, -2. \\ y = \ \ 1, 2, -2, -1. \end{array} \right.$

4. ... $\left\{ \begin{array}{l} xy = 2 \\ x - \ xy \ + y \ = 1 \end{array} \right\} \qquad \left\{ \begin{array}{l} x = \ \ 2, 1. \\ y = \ \ 1, 2. \end{array} \right.$

This example gives only two systems of values. The cause of this is, that if either x or y be eliminated, the resulting equation is only a quadratic.

The most general form of the equations in this case is

$$a(x^2 + y^2) + bxy + c(x + y) = d,$$

and
$$a'(x^2 + y^2) + b'xy + c'(x + y) = d'.$$

Substituting in these equations $u + v$ for x, and $u - v$ for y, they assume the forms

$$mu^2 + nv^2 + pu = q,$$

and
$$m'u^2 + n'v^2 + p'u = q'.$$

Eliminating v^2 from this equation by (262), the resulting equation is a quadratic containing u, from which its value may be found. The value of v is then easily found from either of these last two equations; and hence those of x and y are known. This method of substituting the sum and difference of two quantities for the unknown quantities, may be employed in solving equations higher than quadratics, when there are two equations, and one of them contains only the sum of the two unknown quantities, and the other the sum of their cubes, fourth powers, or fifth powers.

294. The solution of equations which differ in form from those of the preceding cases, must be solved, if possible, by particular methods or analytical artifices, for the discovery of which the student must exercise his own ingenuity. Many of the preceding exercises and examples may be solved by particular methods, and in some cases more expeditiously than by the rule. The following examples will exhibit some of the expedients to which recourse may be had when necessary:—

1. Find the values of x and y in the equations

$$x^2 + y^2 = 25 \quad \ldots \quad [1],$$
$$xy = 12 \quad \ldots \quad [2].$$

Iultiplying [2] by 2, and adding the result to [1], we find

$$x^2 + 2xy + y^2 = 49.$$

'he first member is a complete square; hence, taking the square oot of both,

$$x + y = \pm 7 \quad \dots \quad [3].$$

\gain, subtracting twice [2] from [1], there remains

$$x^2 - 2xy + y^2 = 1,$$

r

$$(x - y)^2 = 1,$$

$$\therefore x - y = \pm 1 \quad \dots \quad [4];$$

dding [4] to [3],

$$2x = \pm 7 \pm 1 = 8, -6, 6, -8;$$

ubtracting [4] from [3],

$$2y = \pm 7 \mp 1 = 6, -8, 8, -6;$$

ience $\qquad x = 4, -3, 3, -4,$

nd $\qquad y = 3, -4, 4, -3.$

2. Find the values of x and y in the equations

$$x^2 + y^2 = 13 \quad \dots \quad [1],$$

$$2xy - x - y = 7 \quad \dots \quad [2].$$

\dding [1] and [2],

$$x^2 + 2xy + y^2 - x - y = 20,$$

r

$$(x + y)^2 - (x + y) = 20;$$

et $x + y = z$, then $z^2 - z = 20$;

ience

$$z = \pm \frac{9}{2} + \frac{1}{2} = 5 \text{ or } -4,$$

r

$$x + y = 5 \text{ or } -4 \quad \dots \quad [3],$$

$$\therefore (x + y)^2 = x^2 + 2xy + y^2 = 25 \text{ or } 16;$$

aking this from twice [1],

$$x^2 - 2xy + y^2 = 1 \text{ or } 10;$$

xtracting the square root,

$$x - y = \pm 1 \text{ or } \pm \sqrt{10} \quad \dots \quad [4];$$

adding [4] to [3], $\quad 2x = 5 \pm 1 \text{ or } \pm \sqrt{10} - 4;$

subtracting [4] from [3],

$$2y = 5 \mp 1 \text{ or } \mp \sqrt{10} - 4;$$

hence the values are $\quad x = 3, 2, \text{ or } \pm \frac{1}{2}\sqrt{10} - 2,$

and $\quad y = 2, 3, \text{ or } \mp \frac{1}{2}\sqrt{10} - 2.$

In solving these equations, the first object is to obtain a quadratic or a simple equation containing only one unknown quantity. This unknown quantity is either one of those in the given equations, or some combination of both of them, as $x + y$ in the second step of the last example. In the latter case, it will be convenient generally to assume a new auxiliary unknown quantity, instead of the combination. When one of the original unknown quantities is found, the other may be easily obtained either from one of the given equations or from some other.

295. The general solution of determinate quadratic equations containing two unknown quantities, can be effected only by the method of elimination, the exposition of which is not adapted to this treatise. It may be easily proved, however, that their solution depends ultimately on that of an equation of the fourth degree containing one unknown quantity. The most general form of the preceding equations is—

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$
$$A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0,$$

which may be expressed in this form:—

$$Ax^2 + (By + D)x + (Cy^2 + Ey + F) = 0,$$
$$A'x^2 + (B'y + D')x + (C'y^2 + E'y + F') = 0.$$

If the coefficients of x^2 be equalised by multiplying the former equation by A', and the latter by A, and the former product be subtracted from the latter, the remainder is—

$$[(AB' - A'B)y + AD' - A'D]x + (AC' - A'C)y^2,$$
$$+ (AE' - A'E)y + AF' - A'F = 0,$$

from which

$$x = \frac{(A'C - AC')y^2 + (A'E - AE')y + (A'F - AF')}{(AB' - A'B)y + (AD' - A'D)};$$

and if this value be substituted for x in either of the given equations, the resulting equation will be evidently of the fourth degree, in terms of y. This equation would give four values of y (286); and hence four corresponding values of x would be obtained.

QUESTIONS PRODUCING QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

1. The sum of two numbers is $= 10$, and their product is $= 24$; required these numbers?

Let $x =$ one of the numbers,

and $y =$ the other,

then, by the question, $x + y = 10$ ... [1],

and $xy = 24$ ... [2],

by [1], $y = 10 - x$ ... [3];

ubstituting this value of y in [2], it becomes

$$x(10 - x) = 24,$$

nd from this equation, which is a quadratic, is found

$$x = \pm 1 + 5 = 6 \text{ or } 4;$$

ence by [3] $\quad y = 10 - 6 \text{ or } 10 - 4 = 4 \text{ or } 6.$

The two systems of values are therefore $\begin{cases} x = 6, 4, \\ y = 4, 6. \end{cases}$

Had x been restricted to represent the greater, and y therefore he less, there would have been but one system—namely, $x = 6$, $= 4$. The solution of the above question is performed by ASE I.

2. The ratio of two numbers is that of 2 to 3, and the difference f their squares is $= 80$: required the numbers.

Let $x =$ the greater,

and $y = \ldots$ less,

hen $\quad x : y = 3 : 2 \quad \ldots \quad [1],$

nd $\quad x^2 - y^2 = 80 \quad \ldots \quad [2],$

y [1], $\quad 3y = 2x \text{ or } y = \frac{2}{3}x;$

ubstituting this value of y in [2], $\quad x^2 - \frac{4}{9}x^2 = 80,$

rom which is found $\quad x = \sqrt{144} = \pm 12;$

ence $\quad y = \frac{2}{3}x = \pm 8.$

3. Find two numbers such that the sum of their products by the respective numbers a and b may be $2s$, and their product equal to a number p.

Let the numbers be x and y,

then $\quad ax + by = 2s \quad \ldots \quad [1],$

and $\quad xy = p \quad \ldots \quad [2],$

by [1], $\quad y = \dfrac{2s - ax}{b} \quad \ldots \quad [3];$

substituting this value of y in [2], and reducing the result, it becomes

$$ax^2 - 2sx = -bp;$$

whence $\quad x = \frac{s}{a} \pm \frac{1}{a}\sqrt{(s^2 - abp)} \text{ or } \frac{1}{a}\{s \pm \sqrt{(s^2 - abp)}\};$

substituting this value of x in [3], and reducing, we find

$$y = \frac{s}{b} \pm \frac{1}{b}\sqrt{(s^2 - abp)} \text{ or } \frac{1}{b}\{s \pm \sqrt{(s^2 - apb)}\}.$$

4. The equirational mean of two numbers is $= 18$, and the difference of their square roots is $= 3$: required the numbers.

Let $x =$ the less, and $y =$ the greater,

then $\sqrt{xy} =$ their equirational mean (431);

and hence $\sqrt{xy} = 18$ ... [1],

also $\sqrt{y} - \sqrt{x} = 3$ ... [2];

squaring these equations, they become

$$xy = 324 \quad \ldots \quad [3],$$

$$y + x - 2\sqrt{xy} = 9;$$

substituting from [1] for $\sqrt{xy}$, we find

$$y + x - 36 = 9, \; y = 45 - x \quad \ldots \quad [4];$$

substituting for y in [3], $45x - x^2 = 324$,

from which $x = \pm\frac{27}{2} + \frac{45}{2} = 36$ or 9;

and by [4], $y = 45 - x = 9$ or 36.

As x is restricted to the less, the solution is the system of values $x = 9$, and $y = 36$.

5. There are three numbers in equirational progression; the sum of the first and second exceeds the third by 1, and 3 times the second is equal to twice the third: what are the numbers?

Let $x =$ the first,

and $y =$ ... second,

then (428), $x : y = y : \frac{y^2}{x}$ the third number;

hence $x + y = \frac{y^2}{x} + 1$ or $x^2 + xy - y^2 - x = 0$ ... [1],

and $3y = \frac{2y^2}{x}$ or $3xy - 2y^2 = 0$ ... ... [2];

dividing [2] by y, $3x - 2y = 0$, and $y = \frac{3}{2}x$ ... ... [3];

substituting this value of y in [1], we find

$$x^2 + \frac{3}{2}x^2 - \frac{9}{4}x^2 - x = 0;$$

learing of fractions, and dividing by x, we obtain

$$x = 4\,;\ \text{hence } y = 6, \text{ and } \frac{y^2}{x} = 9\,;$$

he numbers are therefore 4, 6, and 9.

6. There are four numbers in equidifferent progression; the roduct of the extremes is = 28, and that of the means = 60 : what re the numbers?

$$\begin{aligned} \text{Let } x - y &= \text{second,} \\ x + y &= \text{third,} \end{aligned}$$

hen $\quad 2y = \text{common difference;}$

ence $\quad x - 3y = \text{first,}$

nd $\quad x + 3y = \text{fourth.}$

3y the question,

$$(x - 3y)(x + 3y) = 28 \text{ or } x^2 - 9y^2 = 28 \quad \ldots \quad [1],$$

nd $(x - y)(x + y) = 60$ or $x^2 - y^2 = 60 \quad \ldots \quad [2]$;

ubtracting [1] from [2], $\quad 8y^2 = 32,$

$$y = \sqrt{4} = \pm 2,$$

by [2], $\quad x^2 = 60 + y^2 = 64, \ x = \pm 8.$

Hence the numbers are $x - 3y = 8 - 6 = 2$, $x - y = 6$, $x + y = 10$, and $x + 3y = 14$ or 2, 6, 10, and 14.

7. There are three numbers in harmonic proportion; the sum of the third, with double the first, is = three times the second; and the sum of the squares of the first and third is = 180: required the numbers.

$$\begin{aligned} \text{Let } x &= \text{the first,} \\ \text{and } y &= \ldots \text{ third,} \end{aligned}$$

then $\quad \frac{2xy}{x+y} = \ldots \text{ second (438);}$

and by the question, $\quad 2x + y = \frac{6xy}{x+y} \quad \ldots \quad [1];$

also, $\quad x^2 + y^2 = 180 \quad \ldots \quad [2],$

by [1] $\quad 2x^2 + y^2 - 3xy = 0 \quad \ldots \quad [3],$

the equations [2] and [3] come under CASE II.; and hence assuming $y = vx$, they become

$$x^2 + v^2x^2 = 180, \text{ and } x^2 = \frac{180}{1 + v^2},$$

$$2x^2 + v^2x^2 - 3vx^2 = 0,$$

or $\quad 2 + v^2 - 3v = 0;$

from this quadratic (see 2d EXERCISE, CASE II.) is found

$$v = 2 \text{ or } 1;$$

hence $$x^2 = \frac{180}{1+v^2} = 36,\ y^2 = 180 - 36 = 144,$$

and $$x = 6,\ y = 12,\ \frac{2xy}{x+y} = \frac{144}{18} = 8;$$

the numbers are therefore 6, 8, and 12.

The value of $v = 1$ gives $3\sqrt{10}$ for the three numbers.

8. A and B depart from the same place, in the same direction, and travel at a uniform rate. A sets out 6 hours before B, and B, after travelling $52\frac{1}{2}$ miles, overtakes A; but had their rate of travelling been half a mile per hour less, B would, after travelling only 36 miles, have overtaken A: what was their rate of travelling?

Let $x =$ A's rate of travelling, or the number of miles he travelled per hour, and

$y =$ B's rate.

Then the number of hours that A travels $= 52\frac{1}{2} \div x = \dfrac{105}{2x}$.

and ... ... B ... $= 52\frac{1}{2} \div y = \dfrac{105}{2y}$.

But had they travelled $\frac{1}{2}$ mile per hour less,

then A would have travelled $= 36 \div (x - \frac{1}{2}) = \dfrac{72}{2x-1}$ hours

and B ... ... $= 36 \div (y - \frac{1}{2}) = \dfrac{72}{2y-1}$ ...

But A travels 6 hours longer than B,

hence $$\frac{105}{2x} = \frac{105}{2y} + 6, \text{ or } 210y - 210x - 24xy = 0,$$

and $$\frac{72}{2x-1} = \frac{72}{2y-1} + 6, \text{ or } 156y - 132x - 24xy = 6;$$

dividing each of these equations by 6, they become

$$35y - 35x - 4xy = 0 \quad \ldots \quad [1],$$

$$26y - 22x - 4xy = 1 \quad \ldots \quad [2];$$

subtracting [2] from [1],

$$9y - 13x = -1 \text{ or } y = \frac{13x-1}{9} \quad \ldots \quad [3];$$

substituting this value of y in [1], it becomes, after reduction,

$$52x^2 - 144x = -35;$$

from which quadratic is found $x = 2\frac{1}{2}$ or $\frac{7}{26}$;

and hence by [3], $y = 3\frac{1}{2}$ or $\frac{5}{18}$.

The system of values $x = \frac{5}{2}$, $y = \frac{7}{2}$, satisfy the two conditions of the question literally; but the other systems $x = \frac{7}{26}$, $y = \frac{5}{18}$, give negative results in verifying them for the second condition; so that in order to fulfil this condition, A and B must be supposed to travel in a direction opposite to that of their first journey, and B to depart 6 hours before A.

9. The equidifferent mean of two numbers exceeds their equirational mean by 5, and the latter exceeds their harmonic mean by 4: what are the numbers?

Let x = the less, and y = the greater number,

then their equidifferent mean is $= \frac{1}{2}(x + y)$,

and ... equirational ... $= \sqrt{xy}$,

and ... harmonic ... $= \frac{2xy}{x+y}$;

hence $\frac{1}{2}(x + y) = \sqrt{xy} + 5$ ... [1],

and $\frac{2xy}{x+y} = \sqrt{xy} - 4$ ... [2];

multiplying [1] by [2], $xy = xy + \sqrt{xy} - 20$,

or $\sqrt{xy} = 20$, and $\therefore xy = 400$ [3].

Subtracting [2] from [1], and reducing

$$(x + y)^2 - 18(x + y) = 4xy = 1600 \text{ by [3]}.$$

Let $x + y = z$, then $z^2 - 18z = 1600$;

and hence is found $z = \pm 41 + 9 = 50$ or -32;

whence $x + y = 50$ ... [4],

squaring, $x^2 + 2xy + y^2 = 2500$;

taking 4 times [3] from this equation,

$$x^2 - 2xy + y^2 = 900;$$

hence $x - y = 30$ ... [5],

and [4] + [5] gives $2x = 80$, or $x = 40$;

also [4] − [5] ... $2y = 20$, or $y = 10$.

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The numbers are therefore 10 and 40, and their equidifferen equirational, and harmonic means, are respectively 25, 20, and 1

EXERCISES.

1. The difference between two numbers is $= 7$, and the differenc of their squares is $= 119$: required the numbers, . $= 12$ and

2. The ratio of two numbers is that of 2 to 3, and the differenc of their squares is $= 20$: required the numbers, $= 4$ and

3. The sum of 6 times the greater of two numbers, and 5 time the less, is $= 50$, and their product is $= 20$: what are th numbers? . . . . $= 5$ and

4. The product of a certain number consisting of two places b the sum of its digits is $= 160$, and if it be divided by 4 times th digit in the place of units, the quotient shall be $= 4$: require the number, ... $= 3$

5. If a certain number consisting of two places be divide by the product of its digits, the quotient is $= 2$, and if 27 b added to it, the digits are in an inverted order: what is th number? . . . . $= 3$

6. There are two numbers, the sum of whose squares exceed twice their product by 4, and the difference of their squares exceed half their product by 4: what are the numbers? $= 6$ and

7. There are two numbers whose sum is $= 14$, and if the squa of each be divided by the other, the sum of the quotients is $= 15\frac{1}{6}$ required the numbers, ... $= 6$ and

8. The sum of four numbers in equidifferent progression $= 28$, and the sum of their squares is $= 216$: what are th numbers? . . . . $= 4, 6, 8,$ and 1

9. The equirational mean of two numbers is $= 12$, and the su of their square roots is $= 8$: required the numbers, $= 4$ and 3

10. Three numbers are in equirational progression; the thi exceeds the sum of the first and second by 2, and the third equal to twice the second: what are the numbers? $= 2, 4,$ and

11. There are four numbers in equidifferent progression; t product of the extremes is $= 27$, and of the means is $= 35$: require the numbers, ... $= 3, 5, 7,$ and

12. There are three numbers in harmonic proportion; t second exceeds the first by 3, and the third exceeds twice t second by 4: required the numbers, . . $= 5, 8,$ and 2

13. Three numbers are in harmonic proportion; the sum of t extremes exceeds twice the mean by 6, and the product of t extremes is $= 108$: required the numbers, $= 6, 9,$ and 1

14. Two persons (A and B) depart from the same place, in the same direction, and travel at a uniform rate. A starts 2 hours before B, and, after travelling 30 miles, B overtakes A; but had each of them travelled half a mile more per hour, B would have travelled 42 miles before overtaking A: at what rate did they travel? . . . . . $A = 2\frac{1}{2}$, and $B = 3$ miles, per hour.

15. From two towns (C and D), distant 396 miles, two persons (A and B), setting out at the same time, met each other, after travelling as many days as are equal to the difference of the number of miles they travelled per day; and it appeared that A had travelled 216 miles: how many miles did each travel per day? . . . . $A = 36$, and $B = 30$.

16. A wine-merchant sold 7 dozen of sherry and 12 dozen of claret for £50. Of sherry he sold 3 dozen more for £10 than he did of claret for £6: what was the price of each? = £2 and £3.

RATIO.

296. As quantities in algebra are represented by letters, which denote the values of these quantities in numbers, the following theorems respecting ratios refer merely to the ratios of numbers.*

DEFINITIONS.

297. The *ratio* of one quantity to another is the number of times that the former contains the latter.

Thus, the ratio of 8 to 2 is 4; of 3 to 12 is $\frac{1}{4}$. So the ratio of $14a$ to $7a$ is 2; of $6a^2$ to $2a$ is $3a$.

The ratio of two quantities as of m to n is denoted thus, $m \div n$, or $m : n$, or $\dfrac{m}{n}$; $m : n$ is enunciated m to n, or the ratio of m to n. If m contain n, r times, then $m \div n = r$, or $\dfrac{m}{n} = r$, or $m : n = r$, or $m : n = r : 1$, for $r : 1$ is $= \dfrac{r}{1} = r$.

298. The ratio of two numbers may be either a *terminate* or an *interminate* number (194).

* The usual theorems respecting the ratios of geometrical magnitudes and quantities, generally, are given in the fifth and additional fifth books of the volume on *Plane Geometry* of Chambers's Educational Course.

Thus, $\frac{12}{4} = 3$, an integer; $\frac{8}{12} = \frac{2}{3}$ a fraction; or $\frac{8}{12} = \cdot\dot{6}$ a repeater; $\frac{9}{7} = 1\cdot\dot{2}8571\dot{4}$ a circulate; or $\sqrt{5} : 2 = \frac{\sqrt{5}}{2} = \frac{2\cdot236 \ldots}{2}$ $= 1\cdot118 \ldots$ which is an interminate number, for $\sqrt{5} = 2\cdot236 \ldots$ is such a number.

299. The first term of a ratio is called the *antecedent;* and the second, the *consequent.*

Thus, in the ratio $6:2$, 6 is the antecedent, and 2 the consequent.

300. A ratio is said to be a ratio of *equality*, *majority*, or *minority* according as the antecedent is equal to, or greater or less than the consequent.

301. The antecedents of two or more ratios are called *homologous* terms, and so are the consequents.

302. Ratios are said to be *compounded* when their homologous terms are multiplied together; and the ratio of the two products is said to be *compounded* of the simple ratios, and is hence called a *compound* ratio.

The compound ratio of two simple ratios may be conveniently expressed thus, $(a:b,\ c:d)$ where $a:b$ and $c:d$ are the simple ratios, and $(a:b,\ c:d) = \frac{ac}{bd}$, the compounded ratio.

303. The ratios of the squares of two quantities are said to be *duplicate* of the ratios of the quantities themselves; and that of their cubes, *triplicate;* so the ratios of their square roots are said to be *subduplicate*, and that of their cube roots *subtriplicate* of their ratio.

304. A *multiple* of a quantity is a quantity that contains it exactly.

Thus, $4a$ is a multiple of a by 4, and mb is a multiple of b which contains it m times.

305. A *submultiple* of a quantity is an *aliquot* part or *measure* of that quantity.

Thus, a is a submultiple of ma.*

306. *Equimultiples* of quantities are multiples that contain them the same number of times.

* The terms multiple, equimultiple, submultiple, and equisubmultiple, are in geometry applied to multiples or submultiples by an integer; but in analytic mathematics, the meaning of the first two terms is extended, so as to comprehend multiples by any terminate or interminate numbers.

Thus, $4a$ and $4b$ are equimultiples of a and b by 4; so ma, mb, and mc, are equimultiples of a, b, and c, by m.

307. *Equisubmultiples* of quantities are equal measures or equal aliquot parts of them.

Thus, a and b are equisubmultiples of $4a$ and $4b$; so a, b, and c, are equisubmultiples or equal measures of ma, mb, and mc.

THEOREMS.

308. I. The ratio of two quantities is equal to that of their equimultiples.

Let a, b, be two quantities, and m any number, then $\frac{a}{b} = \frac{ma}{mb}$ (127), where m may be either a terminate or an interminate number.

Cor. The ratio of two quantities is equal to that of their submultiples.

Thus, if $a = nc$, and $b = nd$, then $\frac{a}{b} = \frac{nc}{nd} = \frac{c}{d}$, where c and d are equisubmultiples of a and b by n.

309. II. If the same quantity be added to the terms of a ratio of equality, of majority or of minority, the first will be unaltered, the second diminished, and the third increased.

If a and b be any two quantities, then $\frac{a}{a} = \frac{a+b}{a+b}$, for each is $= 1$, or the ratio of equality $\frac{a}{a}$ is unaltered.

Let c be a third quantity; then reducing the ratios $\frac{a}{b}$, $\frac{a+c}{b+c}$, to the same denominator, they become respectively

$$\frac{ab + ac}{b(b+c)} \text{ and } \frac{ab + bc}{b(b+c)}.$$

Now, when $\frac{a}{b}$ is a ratio of majority, $a > b$, and therefore $ac > bc$, and the former fraction exceeds the latter, or $\frac{a}{b} > \frac{a+c}{b+c}$.

Again, when $\frac{a}{b}$ is a ratio of minority, $a < b$, and therefore $ac < bc$, and the former fraction is less than the latter, or $\frac{a}{b} < \frac{a+c}{b+c}$.

310. III. A ratio of equality is unaltered, a ratio of majority is increased, and a ratio of minority is diminished, by subtracting the same quantity from its terms.

For $\frac{a}{a} = \frac{a-b}{a-b} = 1$.

And when $a > b$, $ac > bc$, and $ab - ac$, is less than $ab - bc$ and hence $\frac{a}{b} < \frac{a-c}{b-c}$, as is evident when they are reduced to th same denominator.

Again, when $a < b$, $ac < bc$, and $db - ac$ is greater thar $ab - bc$; and hence $\frac{a}{b} > \frac{a-c}{b-c}$.

311. IV. Any ratio compounded with a ratio of majority, i increased; and with a ratio of minority, is diminished.

Let $\frac{a}{b}$ be any ratio, and $\frac{c}{d}$ another, the former ratio compounde with the latter, gives $\frac{ac}{bd}$.

The ratio $\frac{a}{b}$ is $= \frac{ad}{bd}$.

When $\frac{c}{d}$ is a ratio of majority, $c > d$; therefore $ac > ad$, an hence $\frac{ac}{bd} > \frac{ad}{bd}$ and the ratio $\frac{ac}{bd} > \frac{a}{b}$.

When $\frac{c}{d}$ is a ratio of minority, $c < d$; therefore $ac < ad$, an hence the ratio $\frac{ac}{bd} < \frac{a}{b}$.

312. V. If a ratio of majority, and another of minority, b compounded, the compound ratio is intermediate in magnitude.

Let $\frac{a}{b}$ be a ratio of majority, and $\frac{c}{d}$ one of minority, then $\frac{ac}{bd} <$ and $> \frac{c}{d}$.

For $\frac{a}{b}$ being compounded with the ratio of minority, is dimi nished (311), or $\frac{ac}{bd} < \frac{a}{b}$.

And $\frac{c}{d}$ being compounded with a ratio of majority, is increase (311), or $\frac{ac}{bd} > \frac{c}{d}$.

313. VI. If the homologous terms of two ratios be adde together, the ratio of the sums is of intermediate magnitude.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the ratios, the former being the greater, then $\frac{a+c}{b+d} < \frac{a}{b}$ and $> \frac{c}{d}$.

For $\frac{a}{b} = \frac{ad}{bd}$, and $\frac{c}{d} = \frac{bc}{bd}$, but $\frac{a}{b} > \frac{c}{d}$; hence $ad > bc$.

Reducing $\frac{a}{b}$ and $\frac{a+c}{b+d}$ to the same denominator, they become

$$\frac{ab+ad}{b(b+d)} \text{ and } \frac{ab+bc}{b(b+d)};$$

but as $ad > bc$, the numerator of the former fraction exceeds that of the latter; and hence $\frac{a}{b} > \frac{a+c}{b+d}$.

Again reducing $\frac{a+c}{b+d}$ and $\frac{c}{d}$ to the same denominator, they become

$$\frac{ad+cd}{(b+d)d} \text{ and } \frac{bc+cd}{(b+d)d};$$

and since $ad > bc$, the numerator of the former fraction exceeds that of the latter, and hence $\frac{a+c}{b+d} > \frac{c}{d}$.

314. VII. The duplicate ratio of two quantities is equal to the ratio of these quantities compounded with itself.

Let a, b, be two quantities, then $a^2 : b^2$ is the ratio of $a : b$ compounded with itself, or with $a : b$ (302); or $(a : b,\ a : b) = a^2 : b^2$.

315. VIII. The triplicate ratio of two quantities is equal to a ratio compounded of three ratios, each of which is equal to the ratio of the quantities.

Let a, b, be two quantities; then the ratio compounded of the three identical ratios $a : b$, $a : b$, and $a : b$, is $a^3 : b^3$ (302), or $(a : b,\ a : b,\ a : b) = a^3 : b^3$.

316. IX. The ratio of two quantities is equal to their subduplicate ratio compounded with itself.

Let $\sqrt{a} : \sqrt{b}$ be the subduplicate ratio of two quantities a and b; then $\sqrt{a} : \sqrt{b}$, compounded with $\sqrt{a} : \sqrt{b}$, gives $a : b$, or $(\sqrt{a} : \sqrt{b},\ \sqrt{a} : \sqrt{b}) = a : b$.

317. X. The ratio of two quantities is equal to the ratio which is compounded of three ratios, each of which is equal to their subtriplicate ratio.

Let $\sqrt[3]{a} : \sqrt[3]{b}$ be the subtriplicate ratio of a and b; then the three identical ratios $\sqrt[3]{a} : \sqrt[3]{b}$, $\sqrt[3]{a} : \sqrt[3]{b}$, and $\sqrt[3]{a} : \sqrt[3]{b}$, being compounded (302), give the ratio $a : b$, or $(\sqrt[3]{a} : \sqrt[3]{b}, \sqrt[3]{a} : \sqrt[3]{b}, \sqrt[3]{a} : \sqrt[3]{b}) = a : b$.

318. XI. The ratio of the first of any number of quantities to the last is equal to the ratio which is compounded of that of the first to the second, of the second to the third, of the third to the fourth, and so on to the last.

For let a, b, c, d, e, be five quantities, then $(a : b,\ b : c,\ c : d,\ d : e)$ $= \frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{d} \cdot \frac{d}{e} = \frac{a}{e}$.

The preceding eleven theorems may all be illustrated by numerical examples.

Thus, to exemplify Theorem VI., let the two ratios be $\frac{5}{6}$ and $\frac{3}{4}$, then $a = 5$, $b = 6$, $c = 3$, and $d = 4$; therefore $a + c = 8$, and $b + d = 10$, or $\frac{a + c}{b + d} = \frac{8}{10} = \frac{4}{5}$. Now, reducing $\frac{5}{6}$ and $\frac{3}{4}$ to a common denominator, they become $\frac{10}{12}$ and $\frac{9}{12}$; hence $\frac{5}{6} > \frac{3}{4}$. And therefore $\frac{4}{5}$ ought to be $< \frac{5}{6}$ and $> \frac{3}{4}$. To verify this, let $\frac{5}{6}$ and $\frac{4}{5}$ be reduced to the same denominator, they become $\frac{25}{30}$ and $\frac{24}{30}$; hence $\frac{5}{6} > \frac{4}{5}$. Again, reducing $\frac{4}{5}$ and $\frac{3}{4}$ to the same denominator, they become $\frac{16}{20}$ and $\frac{15}{20}$; and hence $\frac{4}{5} > \frac{3}{4}$. The other theorems may be similarly illustrated, by giving the letters particular values.

PROPORTION.

DEFINITIONS.

319. A *proportion* or *analogy* consists of two equal ratios, and therefore of four terms.

Let the ratio $a : b$ be equal to that of $c : d$, then $a : b = c : d$ is a proportion, which is sometimes expressed thus, $a : b :: c : d$, and is enunciated thus: the ratio of a to b is equal to that of c to d; or, a is to b as c is to d.

320. The terms of equal ratios, taken in order, are said to be *proportional*, and they are called *proportionals*, or *proportional quantities.*

Thus, if $a : b = c : d$, the terms a, b, c, d, are said to be proportional, and are called proportionals. So if $a : b = c : d = e : f$, a, b, c, d, e, and f, are said to be proportional quantities, or proportionals.

321. The first and last terms of a proportion are called *extremes*, and the second and third *means.*

Thus, if $a : b = c : d$, a and d are called the extremes, and b and c the means.

322. When the first of any number of quantities has to the second the same ratio as the second to the third, and the second to the third the same ratio as the third to the fourth, and so on, the quantities are said to be in *continued proportion*, and are called *continual proportionals.*

Thus, if $a : b = b : c = c : d$, a, b, c, and d, are in continued proportion. So are 2, 4, and 8, for $2 : 4 = 4 : 8$.

323. When three quantities are in continued proportion, the first and third are called *extremes*, and the second a *mean proportional.*

Let $a : b = b : c$, then a and c are the extremes, and b is a mean proportional between them.

324. Four quantities are said to be *directly* proportional, when the first is to the second as the third to the fourth.

Thus, a, b, c, and d, are directly proportional when $a : b = c : d$. Likewise 3, 4, 6, and 8, are directly proportional, for $3 : 4 = 6 : 8$.

Four quantities are said to be *inversely* proportional, when the first is to the second as the fourth to the third.

Thus, a, b, d, and c, are indirectly proportional when $a : b = c : d$. So are 3, 12, 32, and 8, for $3 : 12 = 8 : 32$.

325. Two quantities are said to be *reciprocally* proportional to other two, when one of the former is to one of the latter, as the remaining one of the latter to that of the former; that is, when the former are taken for extremes, and the latter for means, or conversely.

Thus, a, d, are reciprocally proportional to b and c, when $a : b = c : d$, or when $a : c = b : d$. So 2 and 15 are reciprocally proportional to 10 and 3, for $2 : 10 = 3 : 15$, or $2 : 3 = 10 : 15$.

326. Two or more analogies are said to be *compounded* when their corresponding ratios are compounded; that is, by taking the products of their corresponding terms.

327. The terms of an analogy may undergo various changes in respect to their order or magnitude, and still a proportion exist; to denote these changes, the following technical terms are employed:—

Inversion, when the second term is to the first as the fourth to the third;

Alternation, when the first term is to the third as the second to the fourth;

Composition, when the sum of the first and second terms is to the second, as the sum of the third and fourth to the fourth;

Addition, when the first term is to the sum of the first and second, as the third to the sum of the third and fourth;

Division, when the excess of the first term above the second is to the second, as the excess of the third above the fourth to the fourth;

Conversion, when the first term is to its excess above the second, as the third to its excess above the fourth;

Mixing, when the sum of the first and second terms is to their difference, as the sum of the third and fourth to their difference;

Direct equality, when there are several analogies, and two homologous terms in each are equal to two homologous terms in the following, and it is inferred that the remaining terms of the first analogy are directly proportional to the remaining terms of the last; and

Indirect equality, when there are several analogies, and two terms not homologous in each are equal to two terms not homologous in the following, and it is inferred that the remaining terms of the first are reciprocally proportional to the remaining terms of the last.

AXIOMS.

328. If the antecedents and consequents of two ratios are respectively equal, so are the ratios.

329. If two ratios are equal, and also their antecedents, so are their consequents.

330. If two ratios are equal, and also their consequents, so are the antecedents.

These three axioms may be more concisely stated thus: of these three conditions—the equality of two ratios, of their antecedents, and of their consequents—if any two be given, the third also exists.

331. If one ratio exceeds another, and their consequents are equal, the antecedent of the former exceeds that of the latter.

332. If one ratio exceeds another, and their antecedents are equal, the consequent of the former is less than that of the latter.

333. If the consequents of two unequal ratios are equal, the greater ratio is that which has the greater antecedent.

334. If the antecedents of two unequal ratios are equal, the greater ratio is that which has the less consequent.

335. According as the first term of a proportion is greater than, equal to, or less than the second, the third is greater than, equal to, or less than the fourth.

336. If the first term of a proportion be a multiple of the second by any number, the third is the same multiple of the fourth.

THEOREMS.

337. I. The terms of an analogy are proportional by inversion.

Let $a : b = c : d$, then by inversion $b : a = d : c$.

For since $\frac{a}{b} = \frac{c}{d}$, $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$, or (140) $\frac{b}{a} = \frac{d}{c}$ or $b : a = d : c$.

338. II. The terms of an analogy are proportional by alternation.

Let $a : b = c : d$, then by alternation $a : c = b : d$.

For $\frac{a}{b} = \frac{c}{d}$, and multiplying by $\frac{b}{c}$,

$$\frac{a}{b} \cdot \frac{b}{c} = \frac{c}{d} \cdot \frac{b}{c}, \text{ or } \frac{a}{c} = \frac{b}{d};$$

hence $a : c = b : d$.

339. III. The terms of an analogy are proportional when taken in any regular order.

Let . . .	$a : b = c : d$	...	[1];
By inversion, . .	$b : a = d : c$	...	[2];
... alternation of [1],	$a : c = b : d$	...	[3];
... ... [2],	$b : d = a : c$	...	[4];
... equal ratios in [1],	$c : d = a : b$	...	[5];
... inversion of [4],	$d : b = c : a$	...	[6];
... alternation of [5],	$c : a = d : b$	...	[7];
... ... [6],	$d : c = b : a$	...	[8];

There are thus eight different analogies formed by transposing the terms in regular order.

340. IV. The terms of an analogy are proportional by composition.

Let $a : b = c : d$, then by composition, $a + b : b = c + d : d$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} + 1 = \frac{c}{d} + 1$ or $\frac{a+b}{b} = \frac{c+d}{d}$;

that is, $$a + b : b = c + d : d.$$

341. V. The terms of an analogy are proportional by addition.

Let $a : b = c : d$, then by addition, $a : a + b = c : c + d$.

For $\frac{b}{a} = \frac{d}{c}$ (337); and hence $1 + \frac{b}{a} = 1 + \frac{d}{c}$ or $\frac{a+b}{a} = \frac{c+d}{c}$

therefore $$a + b : a = c + d : c,$$

and by inversion, $$a : a + b = c : c + d.$$

342. VI. The terms of an analogy are proportional by division.

Let $a : b = c : d$, then if $a > b$, by division, $a - b : b = c - d : d$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} - 1 = \frac{c}{d} - 1$ or $\frac{a-b}{b} = \frac{c-d}{d}$;

that is, $$a - b : b = c - d : d.$$

343. Cor. If $b > a$, then $d > c$ (335), and $1 - \frac{a}{b} = 1 - \frac{c}{d}$ or $\frac{b-a}{b} = \frac{d-c}{d}$, or $b - a : b = d - c : d$.

Therefore, generally, $a \sim b : b = c \sim d : d$.

344. VII. The terms of an analogy are proportional by conversion.

Let $a : b = c : d$, then if $a > b$, by conversion, $a : a - b = c : c - d$.

For $\frac{b}{a} = \frac{d}{c}$; hence $1 - \frac{b}{a} = 1 - \frac{d}{c}$ or $\frac{a-b}{a} = \frac{c-d}{c}$,

herefore $a - b : a = c - d : d$,

nd by inversion $a : a - b = c : c - d$.

345. Cor. If $b > a$, then $\frac{b}{a} - 1 = \frac{d}{c} - 1$ or $\frac{b-a}{a} = \frac{d-c}{c}$;

ence $b - a : a = d - c : c$, and

y inversion, $a : b - a = c : d - c$.

Hence, generally, $a : a \sim b = c : c \sim d$.

346. VIII. The terms of an analogy are proportional by mixing.

Let $a : b = c : d$; then if $a > b$, $a + b : a - b = c + d : c - d$.

For by addition, $\frac{a}{a+b} = \frac{c}{c+d}$ ... [1];

y composition and inversion, $\frac{b}{a+b} = \frac{d}{c+d}$ ... [2];

ubtracting the latter from the former,

$$\frac{a-b}{a+b} = \frac{c-d}{c+d},$$

r $a - b : a + b = c - d : c + d$; and hence

y inversion, $a + b : a - b = c + d : c - d$.

347. Cor. When $b > a$, subtract [1] from [2], and $\frac{b-a}{a+b} = \frac{d-c}{c+d}$, whence $a + b : b - a = c + d : d - c$.

Hence, generally, $a + b : a \sim b = c + d : c \sim d$.

348. IX. The product of the extremes of a proportion is equal to that of the means.

Let $a : b = c : d$, then $ad = bc$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $bd \cdot \frac{a}{b} = bd \cdot \frac{c}{d}$, or $ad = bc$.

349. Cor. 1. The fourth term of a proportion is equal to the product of the second and third divided by the first.

For $ad = bc$; and hence $d = \frac{bc}{a}$.

Therefore, when three terms of a proportion are given, the ourth can be found.

350. Cor. 2. If the product of two quantities be equal to th of the other two, these four quantities are reciprocally propo tional.

Let $ad = bc$, then $\frac{ad}{bd} = \frac{bc}{bd}$, or $\frac{a}{b} = \frac{c}{d}$; that is, $a : b = c : d$, alternately, $a : c = b : d$.

351. Cor. 3. When two quantities are reciprocally propo tional to other two, if the former two be made the extreme tern of a proportion, the latter two are means, and conversely.

352. Cor. 4. If three quantities be in continued proportion, tl product of the extremes is equal to the square of the mean.

Let $a : b = b : c$, then $ac = b \,.\, b = b^2$.

353. Hence when the extremes are known, the square of tl mean proportional is easily found, and consequently the mea itself. Also, when the mean proportional and one extreme a known, the other extreme can be found, for $a = \frac{b^2}{c}$, and $c = \frac{b^2}{a}$.

354. Cor. 5. If the product of two quantities be equal to tl square of a third, this third is a mean proportional between then

If $ac = b^2$, then $ac = b \,.\, b$, and $a : b = b : c$ (350).

355. X. If any number of quantities be proportional, o antecedent is to its consequent as the sum of all the anteceden to that of all the consequents.

Let a, b, c, d, e, and f, be proportional, then $a : b = a + c + e$ $+ d + f$.

For $\frac{a}{a} = \frac{b}{b}$; and since $a : b = c : d$, by alternation, $\frac{a}{c} =$

also, $a : b = e : f$, therefore by alternation, $\frac{a}{e} = \frac{b}{f}$; hence t reciprocals of these equals are also equal; that is,

$$\frac{a}{a} = \frac{b}{b}, \frac{c}{a} = \frac{d}{b}, \text{ and } \frac{e}{a} = \frac{f}{b}, \text{ or } \frac{a + c + e}{a} = \frac{b + d + f}{b};$$

hence $\qquad a + c + e : a = b + d + f : b,$

or (339), $\qquad a : b = a + c + e : b + d + f.$

356. XI. If the consequents of two analogies be the same, t sum of the first antecedents is to their consequents, as the sum the other antecedents to their consequents.

Let $a : b = c : d$, and $e : b = f : d$, then $a + e : b = c + f : d$.

For by the former proportion, $\frac{a}{b} = \frac{c}{d}$, and by the second, $\frac{e}{b} = \frac{f}{d}$; hence $\frac{a}{b} + \frac{e}{b} = \frac{c}{d} + \frac{f}{d}$, or $\frac{a+e}{b} = \frac{c+f}{d}$, or $a + e : b = c + f : d$.

357. XII. In any analogy, as one antecedent is to its consequent, so is the sum or difference of the antecedents to that of the consequents.

Let $a : b = c : d$, then if $a > c$, $a : b = a \pm c : b \pm d$.

For (339) $\frac{c}{a} = \frac{d}{b}$, $\therefore 1 \pm \frac{c}{a} = 1 \pm \frac{d}{b}$, or $\frac{a \pm c}{a} = \frac{b \pm d}{b}$; hence (339) $a : b = a \pm c : b \pm d$.

358. XIII. Any equimultiples of the first and second terms of an analogy are proportional to any equimultiples of the third and fourth.

Let $a : b = c : d$, then $ma : mb = nc : nd$.

For since $\frac{a}{b} = \frac{c}{d}$, (127) $\frac{ma}{mb} = \frac{nc}{nd}$, or $ma : mb = nc : nd$.

359. Cor. 1. Any equimultiples of the terms of an analogy are proportional.

For when $n = m$, $ma : mb = mc : md$.

360. Cor. 2. Any equimultiples of two quantities have the same ratio as any other two equimultiples.

For $\frac{ma}{mb} = \frac{na}{nb}$, or $ma : mb = na : nb$.

361. Cor. 3. Any two quantities are proportional to any of their equimultiples.

For $\frac{a}{b} = \frac{ma}{mb}$, or $a : b = ma : mb$.

362. XIV. Any equimultiples of the homologous terms of a proportion are also proportional.

Let $a : b = c : d$, then $ma : nb = mc : nd$.

For by alternation, $a : c = b : d$; therefore (358) $ma : mc = nb : nd$, and by alternation $ma : nb = mc : nd$.

Cor. 1. When $m = 1$, $a : nb = c : nd$.

Cor. 2. When $n = 1$, $ma : b = mc : d$.

363. Cor. 3. According as ma is greater than, equal to, or less than nb, mc is greater than, equal to, or less than nd (335).*

* The principle in this corollary, when m and n are any integral numbers, is the property assumed by Euclid as the definition of the proportionality of four quantities.

The two preceding theorems, and their corollaries, are tru when equal submultiples are taken instead of equimultiples; fo the steps of the demonstrations are true whether m or n b integral or fractional.

364. XV. If any number of analogies have two homologou terms in each equal to two in the following, the remaining term are proportional by direct equality.

Let $a : b = c : d$, and $b : e = d : f$; then by direct equalit $a : c = e : f$.

For (339) by the first analogy, $\frac{a}{c} = \frac{b}{d}$, and by the second, $\frac{b}{d} = \frac{e}{f}$

hence $\frac{a}{c} = \frac{e}{f}$, or $a : c = e : f$.

If there be a third analogy, as $e : g = f : h$, then $\frac{g}{h} = \frac{e}{f} = \frac{a}{c}$, o $a : c = g : h$.

If the three analogies be arranged thus,

$$a : b = c : d,$$
$$b : e = d : f,$$
$$e : g = f : h,$$

then it is evident that the terms, to which none of the rest a equal, are a, c, g, and h, and

$$a : c = g : h.$$

365. XVI. If any number of analogies have two terms n homologous in each, equal to two not homologous in the followin the remaining terms are proportional by indirect equality.

Let $a : b = c : d$, and $b : e = f : c$, then by indirect equalit $a : e = f : d$.

For by the first proportion (348), $ad = bc$, and by the secon $bc = ef$; therefore $ad = ef$, and (350) $a : e = f : d$.

If there be a third proportion, as $e : g = h : f$, then $ef = gh$ therefore $ad = gh$, and $a : g = h : d$.

Arranging the three analogies thus,

$$a : b = c : d,$$
$$b : e = f : c,$$
$$e : g = h : f,$$

the terms to which none of the rest are equal, are a, d, g, and h hence

$$a : g = h : d.$$

366. XVII. Ratios that are compounded of equal ratios ar equal.

Let $a : b = (c : d,\ e : f)$, and let $a' : b' = (c' : d',\ e' : f')$; also let $c : d = c' : d'$, and $e : f = e' : f'$, then is $a : b = a' : b'$.

For $\frac{a}{b} = \frac{c}{d} \cdot \frac{e}{f}$ (302), and $\frac{a'}{b'} = \frac{c'}{d'} \cdot \frac{e'}{f'}$; but since $\frac{c}{d} = \frac{c'}{d'}$, and $\frac{e}{f} = \frac{e'}{f'}$; therefore $\frac{a}{b} = \frac{a'}{b'}$, or $a : b = a' : b'$.

367. XVIII. The reciprocals of the terms of an analogy are proportional.

Let $a : b = c : d$, then $\frac{1}{a} : \frac{1}{b} = \frac{1}{c} : \frac{1}{d}$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{b}{a} = \frac{d}{c}$, or $\frac{1}{a} \div \frac{1}{b} = \frac{1}{c} \div \frac{1}{d}$; that is, $\frac{1}{a} : \frac{1}{b} = \frac{1}{c} : \frac{1}{d}$.

368. XIX. If two analogies be compounded, the resulting terms are proportional.

Let $a : b = c : d$, and $e : f = g : h$, then $ae : bf = cg : dh$.

For $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$; therefore $\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{g}{h}$, or $\frac{ae}{bf} = \frac{cg}{dh}$; that is, $ae : bf = cg : dh$.

The proposition may be proved in the same way, whatever be the number of analogies to be compounded.

369. XX. The same powers of the terms of an analogy are proportional.

Let $a : b = c : d$, then $a^n : b^n = c^n : d^n$.

For $\frac{a}{b} = \frac{c}{d}$, therefore $\left(\frac{a}{b}\right)^n = \left(\frac{c}{d}\right)^n$, or $\frac{a^n}{b^n} = \frac{c^n}{d^n}$, or $a^n : b^n = c^n : d^n$.

370. XXI. The same roots of the terms of an analogy are proportional.

Let $a : b = c : d$, then $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $\sqrt[n]{\frac{a}{b}} = \sqrt[n]{\frac{c}{d}}$, or $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{\sqrt[n]{c}}{\sqrt[n]{d}}$, or $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}$.

371. XXII. Any root of any power of the terms of an analogy are proportional.

Let $a : b = c : d$, then $a^{\frac{m}{n}} : b^{\frac{m}{n}} = c^{\frac{m}{n}} : d^{\frac{m}{n}}$.

For $a^m : b^m = c^m : d^m$; hence (370) $a^{\frac{m}{n}} : b^{\frac{m}{n}} = c^{\frac{m}{n}} : d^{\frac{m}{n}}$.

372. Let $a : b = c : d$, and let m and n be any two numbers, the since $\frac{a}{b} = \frac{c}{d}, \frac{ma}{nb} = \frac{mc}{nd}$, and $\frac{ma}{nb} \pm 1 = \frac{mc}{nd} \pm 1$, or $\frac{ma \pm nb}{nb} = \frac{mc \pm nd}{nd}$ or $\frac{ma \pm nb}{mc \pm nd} = \frac{nb}{nd} = \frac{b}{d}$.

If p and q be any other two numbers, it may be similarly prove that $\frac{pa \pm qb}{pc \pm qd} = \frac{b}{d}$; and hence

$$\frac{ma \pm nb}{mc \pm nd} = \frac{pa \pm qb}{pc \pm qd}, \text{ or } \frac{ma \pm nb}{pa \pm qb} = \frac{mc \pm nd}{pc \pm qd};$$

that is, $\quad ma \pm nb : pa \pm qb = mc \pm nd : pc \pm qd.$

PROPORTIONAL EQUATIONS.

DEFINITIONS.

373. A *proportional equation* is an equation subsisting betwee two variable quantities, expressing merely the proportionality the two members of the equation.

Thus, if x and y are two variable quantities so related that an two values of x, as x' and x'', are proportional to the correspond ing values of y—namely, y' and y''—then $x' : x'' = y' : y''$, whic is concisely expressed thus, $x \propto y$, which expression is called proportional equation; and the sign $\propto$ is read *varies as.*

As a particular example, let x be the rate of travelling of stage-coach, or the number of miles travelled in an hour, an y the number travelled in a day; then if x', x'', the rates travelling for any two days, be 7 and 8 miles an hour respectivel the corresponding distances travelled on these two days, or y' an y'', are 168 and 192; and $7 : 8 = 168 : 192$, or $x' : x'' = y' : y'$ The same proportion exists for any other two rates of travellin and the corresponding number of miles travelled per day, so tha the proportion $x' : x'' = y' : y''$ is true generally for any two rat of travelling; and this is expressed by the proportional equatio $x \propto y$, which always implies the proportionality of *four* quan tities—namely, of any two values of x, and the correspondin values of y.

Innumerable examples of proportional equations occur in mathematical and physical science; quantities in the latter science, like those in the former, being represented by letters which express their numerical values, or the number of times that the assumed unit of measure is contained in them.

374. If two quantities are so connected that when one of them varies, the other varies in the same proportion, the former is said to vary *directly* as the other.

Thus, if x and y are so related that any two values of x, as x' and x'', and the corresponding values of y—namely, y' and y''—are so related that $x' : x'' = y' : y''$, then x and y are said to vary directly, and this relation is expressed by $x \propto y$.

375. If two quantities be so related that when one of them varies, the reciprocal of the other varies in the same proportion, the former is said to vary *inversely* as the latter.

Thus, if $x' : x'' = \frac{1}{y'} : \frac{1}{y''}$, x and y are said to vary inversely, and this is expressed by $x \propto \frac{1}{y}$.

376. If when one quantity varies, the product of other two varies in the same proportion, the former is said to vary as the two latter *jointly*.

Let y', z', and y'', z'', be two systems of values of y and z corresponding to x' and x'', two values of x, then if $x' : x'' = y'z' : y''z''$, x is said to vary as y and z jointly, or $x \propto yz$.

377. If when one quantity varies, the quotient of a second divided by a third also varies in the same proportion, the former is said to vary *directly* as the second, and *inversely* as the third.

Let x', y', z', and x'', y'', z'', be two systems of values of x, y, and z, then if $x' : x'' = \frac{y'}{z'} : \frac{y''}{z''}$, x is said to vary directly as y, and inversely as z, or $x \propto \frac{y}{z}$.

THEOREMS.

378. I. A proportional equation may be converted into an absolute equation, by multiplying one side by some constant quantity.

Let $x \propto y$ be the given equation, then if m be a constant quantity of a proper value, $x = my$.

For if x', y', and x'', y'', be any two systems of values of x and y, $x' : x'' = y' : y''$ (374), therefore (348) $x'' y' = x' y''$; therefore

$$x'' = \frac{x'}{y'} y''.$$

If now x''' and y''' be another system of values of x and y, it may be similarly proved that

$$x''' = \frac{x'}{y'} y'''.$$

And the same relation is proved for any other system; hence if m be taken for $\frac{x'}{y'}$, then generally

$$x = my.$$

Thus, if, as in a former example (373), x be the rate of travelling per hour of a stage-coach, and y the rate per day,

then $\quad x' : y' = 1 : 24$; therefore $\frac{x'}{y'} = \frac{1}{24} = m$,

and $\quad x = my$ is $x = \frac{1}{24} y$.

And this absolute equation is true for any system of values of x and y. Thus, if $x = 10$ miles per hour, then $10 = \frac{1}{24} y$, or $y = 24 \times 10 = 240$ miles per day.

379. Cor. Hence if any system of values of x and y be known, the constant quantity m can be found.

380. II. If one variable quantity be equal to the product of another by a constant quantity, the former variable quantity varies directly as the latter variable quantity.

Let $x = my$, then $x \propto y$.

For $x' = my'$, and $x'' = my''$; therefore $x' : x'' = my' : my'' = y' : y''$; therefore $x \propto y$.

381. III. The reciprocals of the members of a proportional equation also form a proportional equation.

Let $x \propto y$, then $\frac{1}{x} \propto \frac{1}{y}$.

For (378) $x = my$; therefore $\frac{1}{x} = \frac{1}{m} \cdot \frac{1}{y} = n \cdot \frac{1}{y}$, if $n = \frac{1}{m}$; hence (380) $\frac{1}{x} \propto \frac{1}{y}$.

382. IV. If one quantity vary as another, any multiples or parts of them also vary as each other.

Let $x \propto y$, then $mx \propto ny$.

For $x' : x'' = y' : y''$, therefore $mx' : mx'' = ny' : ny''$, or $mx \propto ny$.

And this result is true, whether m and n be integral or fractional.

383. Cor. 1. If one quantity vary as another, it will also vary as any multiple or aliquot part of the other.

For when $m = 1$, $x \propto ny$.

384. Cor. 2. Any multiples or aliquot parts of a variable quantity vary as each other.

Let x be a variable quantity, then $mx \propto nx$.

For $mx' : mx'' = nx' : nx''$, or $mx \propto nx$.

When $m = 1$, $x \propto nx$.

385. V. If both sides of a proportional equation be multiplied or divided by a constant or a variable quantity, the equation will still exist.

Let $x \propto y$, then if z be either a constant or a variable quantity, $xz \propto yz$, and $\frac{x}{z} \propto \frac{y}{z}$.

For $x' : x'' = y' : y''$, and if z', z'', be any two values of z, $x'z' : x''z'' = y'z' : y''z''$, and $\frac{x'}{z'} : \frac{x''}{z''} = \frac{y'}{z'} : \frac{y''}{z''}$ (362); that is, $xz \propto yz$, and $\frac{x}{z} \propto \frac{y}{z}$.

When z is constant, $z' = z'' = z$, and the proof is the same.

386. Cor. 1. When one quantity varies directly as another, their ratio is constant.

Let $x \propto y$, then $\frac{x}{y} \propto 1$.

For $\frac{x}{y} \propto \frac{y}{y} \propto 1$.

That is, $\frac{x'}{y'} : \frac{x''}{y''} = 1 : 1$ or $\frac{x'}{y'} = \frac{x''}{y''}$. Or thus:—By (378) $x = my$, therefore $\frac{x}{y} = m$.

387. Cor. 2. Any variable quantity, which is a factor of one member of a proportional equation, is proportional to the other member divided by the remaining factors of the former member.

Let $xy \propto z$, then $x \propto \frac{z}{y}$.

Let $xyz \propto v$, then $xy \propto \frac{v}{z}$ and $x \propto \frac{v}{yz}$.

Let $xyz \propto \frac{uv}{w}$, then $xy \propto \frac{uv}{wz}$ and $x \propto \frac{uv}{wyz}$.

388. Cor. 3. Any variable quantity, which is a divisor in one member of a proportional equation, is proportional to the other part of this member divided by the other member.

Let $\frac{y}{x} \propto z$, then $x \propto \frac{y}{z}$.

For $\frac{1}{x} \propto \frac{z}{y}$; therefore (381) $x \propto \frac{y}{z}$.

So if $\frac{yz}{x} \propto uv$, $\frac{1}{x} \propto \frac{uv}{yz}$ (385); hence $x \propto \frac{yz}{uv}$.

From these two corollaries, it appears that any factor or divisor of a member of a proportional equation may be made to stand alone; and that one member may be divided by the other, the quotient being a constant, or 1.

389. Cor. 4. If one quantity varies as other two jointly, either of the latter varies as the first directly, and the remaining one inversely.

Let $x \propto yz$, then $y \propto \frac{x}{z}$ or $z \propto \frac{x}{y}$ (385).

390. VI. If one side of a proportional equation be unity, the other side is equal to some constant quantity.

Let $xy \propto 1$, then $xy = m$.

For $x'y' : x''y'' = 1 : 1 = m : m$; and if $x'y' = m$, so is $x''y''$, and generally $xy = m$.

The product of any particular system of values of the variables is = the value of the constant m.

391. VII. If one side of a proportional equation be equal to a constant, the other side will be any constant or unity.

Let $xy \propto m$, then $xy \propto n$, or $xy \propto 1$.

For $x'y' : x''y'' = m : m = n : n = 1 : 1$; hence $xy \propto n$, or $xy \propto 1$.

392. Cor. If the product of two quantities be constant, they vary inversely as each other.

Let $xy = m$, then $xy \propto 1$; therefore (385) $x \propto \frac{1}{y}$, and $y \propto \frac{1}{x}$.

393. VIII. If the first of three quantities varies as the second, and the second as the third, the first varies also as the third.

Let $x \propto y$, and $y \propto z$, then $x \propto z$.

For $x = my$, and $y = nz$ (378); therefore $x = mnz$, and consequently (380) $x \propto z$.

So if $x \propto y$, and $y \propto \frac{1}{z}$, $x \propto \frac{1}{z}$.

394. Cor. 1. If the first of four quantities varies as the second, and the second as the third, and the third as the fourth, the first will also vary as the fourth.

Let $x \propto y$, and $y \propto z$, and $z \propto v$, then $x \propto v$.

For $x \propto z$ (393), and $z \propto v$, therefore $x \propto v$.

395. Cor. 2. If one quantity vary as a second, the second as a third, and so on for any number, each varies as each of the others.

396. IX. If the first of three quantities varies as the second, and the second as the third, the second will vary as the sum or difference of the first and third.

Let $x \propto y$, and $y \propto z$, then $x \pm z \propto y$.

For $x = my$, and $z = ny$; therefore $x \pm z = (m \pm n)y$; and hence (380) $x \pm z \propto y$.

397. Cor. 1. If one quantity varies as another, their sum or difference will vary as the other.

Let $x \propto y$, then $x \pm y \propto y$.

For $x = my$; therefore $x \pm y = (m \pm 1)y$; and hence $x \pm y \propto y$.

398. Cor. 2. If one quantity varies as another, their sum and difference will vary as each other.

Let $x \propto y$, then $x + y \propto x - y$.

For $x = my$; therefore $x + y = (m + 1)y$, and $x - y = (m - 1)y$. But (382) $(m + 1)y \propto (m - 1)y$; therefore $x + y \propto (m + 1)y \propto (m - 1)y \propto x - y$; therefore (394) $x + y \propto x - y$.

399. X. If the members of a proportional equation be respectively multiplied or divided by those of another, the products or quotients will form a proportional equation.

Let $x \propto y$, and $z \propto v$, then $xz \propto yv$.

For $x = my$, and $z = nv$; therefore $xz = mnyv$; and hence $xz \propto yv$.

Also, $\frac{x}{z} = \frac{m}{n} \cdot \frac{y}{v}$; therefore (380) $\frac{x}{z} \propto \frac{y}{v}$.

400. Cor. 1. If the first of three quantities varies as the second, and the second as the third, the square of the second will vary as the first and third jointly.

Let $x \propto y$, and $y \propto z$, then $xz \propto y^2$.

For $x \propto y$, and $z \propto y$, hence $xz \propto y^2$.

401. Cor. 2. If a factor in one of the members of a proportional equation vary also as any other quantity, the latter may be substituted in place of it in the former equation.

Let $x \propto yz$, and $y \propto \frac{u}{v}$, then $x \propto \frac{uz}{v}$.

For (399) $xy \propto \frac{uyz}{v}$; therefore $x \propto \frac{uz}{v}$.

402. XI. If one quantity vary as another, any power or root of the former will vary as the same power or root of the latter.

Let $x \propto y$, then $x^n \propto y^n$, whether n be integral or fractional.

For $x = my$; therefore $x^n = m^n y^n = ay^n$, if $a = m^n$, which is constant. Hence $x^n \propto y^n$.

403. Cor. If one quantity vary as a second, and the second as the third, the sum or difference of the first and third will vary as the square root of their product.

Let $x \propto y$, and $y \propto z$, then $x \pm z \propto \sqrt{xz}$.

For (396) $x \pm z \propto y$, and (400) $y^2 \propto xz$; therefore $y \propto \sqrt{xz}$; and hence $x \pm z \propto \sqrt{xz}$.

404. XII. If four quantities be always proportional, and only the first of them be constant, the fourth will vary as the product of the second and third.

Let $m : x = y : z$, then $z \propto xy$.

For $mz = xy$; hence (380) $z \propto xy$.

405. Cor. If the first and second, or the first and third, are constant, the other two will vary as each other; and if the first and fourth are constant, the second will vary inversely as the third.

Let $m : n = x : y$, then $nx = my$; and hence $x \propto y$ when m and n are constant.

Let $m : x = n : y$, then $nx = my$; and hence $x \propto y$ when m and n are constant.

Let $m : x = y : n$, then $xy = mn$, or $xy \propto 1$; hence $x \propto \frac{1}{y}$ when m and n are constant.

406. XIII. If three quantities be so related that when either the second or third is constant, the first varies as the other; when both the second and third vary, the first will vary as their product.

Let $x \propto y$ when z is constant,

and $x \propto z$ ... y ... ,

then $x \propto yz$ when both x and y vary.

For if x' and x''' be two values of x and y', y'' the corresponding values of y when z is constant, then

$$x' : x''' = y' : y''.$$

But if z be now supposed to vary, then if z' is the value of z that corresponds to the value x''' of x, and if x'', z'', is another system of values of x and z,

$$x''' : x'' = z' : z'';$$

and compounding the preceding proportion with the latter,

$$x' : x'' = y'z' : y''z'',$$

therefore $$x \propto yz.$$

This proposition is exemplified by many theorems in geometry regarding the areas of figures. For instance, if s, b, and h, denote the area, the base, and the height of a rectangle, or parallelogram, or triangle, they have the same relation as x, y, and z, in this proposition.

407. Cor. 1. If one quantity varies as other two jointly, the first will vary as either of the other two when the third is constant.

Let $x \propto yz$, then when z is constant, and $= m$, $x \propto my$, or $x \propto y$; and when y is constant, and $= n$, $x \propto nz$, or $x \propto z$.

The variations of y and z are supposed to be independent of each other in this theorem and corollary, as b and h are in the case of a rectangle or triangle; but when y and z are not independent ot each other, and vary directly as each other, though the theorem then exists, this corollary cannot exist, but the following one is a consequence of the relation between y and z.

408. Cor. 2. If one quantity varies as other two jointly, and if the two latter vary directly as each other, the first will vary as the square of either of the other two.

Let $x \propto yz$, and $y \propto z$, then $x \propto y^2$ or $x \propto z^2$.

For (401) substituting y for z, $x \propto y^2$, or substituting z for y, $x \propto z^2$.

This corollary is exemplified by geometrical theorems respecting the areas of similar figures.

409. Cor. 3. If one quantity varies as each of any number of quantities when the rest are constant, when all are variable, it varies as their product.

EQUIDIFFERENT PROGRESSION.*

410. If the difference between any term of a series of quantities that increases or decreases, and the following term, be always the same, it is called an *equidifferent series* or *progression;* in the former case it is an *increasing*, and in the latter a *decreasing*, series.

The constant difference between any two successive terms is called the *common difference.*

In the following, a is $=$ the first term, $d =$ the common difference, $n =$ the number of terms, $z =$ the last term, and $s =$ the sum of the series.

Thus, 1, 3, 5, 7, &c. is an equidifferent increasing series, the common difference of which is $= 2$. So is the series a, $a + d$, $a + 2d$, ... the common difference of which is d. The series 20, 17, 14, 11, ... is a decreasing series, whose common difference is 3. So is $a + 8d$, $a + 6d$, $a + 4d$, ... the common difference being $2d$. An increasing series may begin with negative terms, as -11, -8, -5 -2, 1, 4, 7, 10, ... and a decreasing one may terminate with negative terms, as 6, 4, 2, 0, -2, -4, -6 ...

THEOREMS.

411. I. The last term of an equidifferent series is equal to the sum or difference of the first term and the product of the common difference by a number which is one less than the number of terms, according as the series is increasing or decreasing.

Let a, $a + d$, $a + 2d$, $a + 3d$, ... be the series; then if the number of terms be n, the last term is evidently $a +$ the product of d by $n - 1$; and hence,

$$z = a + (n - 1)d.$$

For instance, when the last term is the third, $n = 3$, and $n - 1 = 2$, and $z = a + 2d$. When the last term is the fourth, $n = 4$, and $n - 1 = 3$, and $z = a + 3d$.

Let a, $a - d$, $a - 2d$, $a - 3d$, ... be a decreasing series, and it is similarly evident that

$$z = a - (n - 1)d.$$

* By French algebraists, this series is called 'progression par équidifférence,' or simply 'par différence.' Equirational series, treated of in the next section, is called by them 'progression par quotients égeaux,' or merely 'par quotient.' The term *équidifférent* is adopted by some English authors, and I have adopted the new term *equirational*, because the common terms, arithmetical and geometrical, are not appropriate.

hese two results may be expressed thus,

$$z = a \pm (n - 1)d \quad \ldots \quad [1],$$

he upper sign being taken for an increasing, and the lower for a ecreasing, series.

412. Cor. Any term of the series may be found from this ormula.

413. II. The sum of the first and last terms of an equidifferent eries is equal to the sum of any other two terms that are equally istant from the first and last.

Let the series be

$$, a + d, a + 2d, \ldots\ldots\ldots\ldots\ldots\ldots\ldots a + (n - 2)d, a + (n - 1)d;$$

nd let the series be taken also in a reverse order,

$$+ (n - 1)d, a + (n - 2)d, \ldots\ldots\ldots\ldots\ldots\ldots\ldots a + 2d, a + d, a,$$

hen the first term of the former, with that of the latter, is

$$= 2a + (n - 1)d,$$

nd the sum of the second terms, as also of the third and fourth, s evidently equal to the same quantity; and so on to their last erms. But the first terms of the two series are the first and last f the first series; their second terms are the second and last but ne of the first series, and so on; hence the theorem is evident. When the series is decreasing, the proof is similar.

414. Cor. 1. When the series consists of four terms, the sum of he extremes is equal to that of the means.

415. Cor. 2. When the series consists of three terms, the sum f the extremes is equal to twice the mean.

416. Cor. 3. When the number of terms is uneven, the sum of he first and last, or of any two equally distant from them, is equal o twice the middle term.

For in this case the number denoting the place of the middle term is $\frac{n+1}{2}$; and hence this term is $= a + \left(\frac{n+1}{2} - 1\right)d$ $= a + \frac{n-1}{2}d$, and its double is $2a + (n - 1)d$, which is equal to $a + z$.

417. III. The sum of the terms of an equidifferent series is equal to the sum of the first and last terms multiplied by half the number of terms.

For, arranging the series both in a direct and in a reverse order, as in the preceding theorem, the sum of the first terms of the two series becomes (413)

$$= 2a + (n - 1)d = a + z \text{ by } [1],$$

is also the sum of each succeeding pair of terms. But the

number of these pairs of terms is $= n$; and hence their sum is $a + z$, taken n times, or $= n(a + z)$. This, however, is the sum of the two series, or it is twice the sum of the given series; hence,

$$s = \frac{n}{2}(a + z).$$

418. When any three of the five quantities a, z, d, n, and s, are given, the other two may be found from the two equations,

$$[1] \quad \ldots \quad z = a \pm (n - 1)d, \qquad s = \frac{n}{2}(a + z) \quad \ldots \quad [2],$$

which are then two equations containing only two unknown quantities, the values of which may be found by the preceding methods for the solution of two determinate equations.

The equations will be simple in every case except two—namely, when a and n, or z and n, are the unknown quantities. In these two cases the equations will be quadratics, for a and n, or z and n, are in both equations, and their product is in the second.

The cause of the double values of the required terms in these two cases will be easily seen from this circumstance, that in both these cases s is given, and it has the same value for two different series of terms when the progression is carried out to negative terms. For example, the series 11, 9, 7, 5, 3, 1, $-$ 1, $-$ 3, $-$ 5, ... gives for the first *three* terms $s = 27$, and for *nine* terms it also gives $s = 27$; therefore n has two values, 3 or 9, and z has also two values, 7 or $-$ 5.

The number of cases, arranged according to the data, is ten, or the combinations of 5 quantities taken 3 and 3 (456); the cases are:—

	Given.	Sought.		Given.	Sought.
1...	$a, d, n,$ ...	z and s	6...	$d, n, z,$ ...	a and s
2...	$a, d, z,$ ...	n ... s	7...	$d, n, s,$ ...	a ... z
3...	$a, n, z,$ ...	d ... s	8...	$n, z, s,$ ...	a ... d
4...	$a, n, s,$ ...	d ... z	9...	$a, d, s,$ ...	n ... z
5...	$a, z, s,$ ...	d ... n	10...	$d, z, s,$ ...	a ... n

In all these cases, except the two last, one of the unknown quantities is directly found from one of the equations [1] or [2], and the remaining one is then directly found from the other of these two equations. In the 9th case, n is found from [3], and then z from [2]; and in the 10th case, n is found from [4], and then a from [2]. The last two cases are those in which the unknown quantities have two values, which may be found from [1] and [2], independently of [3] and [4]. In the formulæ that have double signs, the upper one is to be taken in the case of an increasing, and the lower in that of a decreasing, series; the solutions are contained in the following table:—

CASES.	GIVEN.	SOUGHT.	FORMULAS.
1.	$d, n, z,$	$a,$	$z - (n-1)d.$
2.	$n, z, s,$	...	$\frac{2s}{n} - z.$
3.	$d, z, s,$	...	$\frac{d}{2} \pm \frac{1}{2}\sqrt{\{(2z+d)^2 - 8ds\}}.$
4.	$d, n, s,$	...	$\frac{s}{n} - \frac{(n-1)d}{2}.$
5.	$a, d, n,$	$z,$	$a + (n-1)d.$
6.	$a, n, s,$	...	$\frac{2s}{n} - a.$
7.	$a, d, s,$	...	$-\frac{d}{2} \pm \frac{1}{2}\sqrt{\{(2a-d)^2 + 8ds\}}.$
8.	$d, n, s,$	...	$\frac{s}{n} + \frac{(n-1)d}{2}.$
9.	$a, n, z,$	$d,$	$\frac{z-a}{n-1}.$
10.	$a, n, s,$	...	$\frac{2(s-an)}{n(n-1)}.$
11.	$a, z, s,$	...	$\frac{z^2-a^2}{2s-a-z}$ or $\frac{(z+a)(z-a)}{2s-a-z}.$
12.	$n, z, s,$	...	$\frac{2(nz-s)}{n(n-1)}.$
13.	$a, d, z,$	$n,$	$1 + \frac{z-a}{d}.$
14.	$a, z, s,$	...	$\frac{2s}{a+z}.$
15.	$a, d, s,$	...	$\frac{1}{2} - \frac{a}{d} \pm \frac{1}{2d}\sqrt{\{(2a-d)^2 + 8ds\}}.$
16.	$d, z, s,$	...	$\frac{1}{2} + \frac{z}{d} \pm \frac{1}{2d}\sqrt{\{(2z+d)^2 - 8ds\}}.$
17.	$a, n, z,$	$s,$	$\frac{n}{2}(a+z).$
18.	$a, d, z,$	...	$\frac{a+z}{2} + \frac{z^2-a^2}{2d}.$
19.	$a, d, n,$	...	$\frac{n}{2}\{2a + (n-1)d\}.$
20.	$d, n, z,$	...	$\frac{n}{2}\{2z - (n-1)d\}.$

EXAMPLES.

1. Given the first term of an increasing equidifferent series 3, the common difference 2, and the number of terms 21: to find the last term and the sum of the series.

Here $a = 3$, $d = 2$, $n = 21$, therefore

$$z = a + (n - 1)d = 3 + (21 - 1)2 = 3 + 20 \times 2 = 3 + 40 = 4$$

and $s = \frac{n}{2}(a + z) = \frac{21}{2}(3 + 43) = \frac{21}{2} \times 46 = 21 \times 23 = 483.$

2. Given the first and last terms of a decreasing equidifferent series 100 and 1 respectively, and the common difference 3: find the number of terms and the sum of the series.

Here $a = 100$, $z = 1$, and $d = 3$; hence by [1],

$$n - 1 = -\frac{z - a}{d} = \frac{a - z}{d} = \frac{100 - 1}{3} = 33;$$

$$\therefore n = 33 + 1 = 34,$$

and $$s = \frac{n}{2}(a + z) = \frac{34}{2}(100 + 1) = 17 \times 101 = 1717$$

3. The first term of a decreasing equidifferent progression is 1 the common difference 2, and the sum of the series 27: require the number of terms and the last term.

This is the 9th case, and here $a = 11$, $d = 2$, and $s = 2$ Substituting the value of z in [1] and that of s in [2] thus,

$$s = \frac{n}{2}\{a + a - (n - 1)d\} = \frac{n}{2}\{2a - (n - 1)d\};$$

$$\therefore 27 = \frac{n}{2}\{2 \times 11 - (n - 1)2\} = \frac{n}{2}(22 - 2n + 2)$$

$$= \frac{n}{2}(24 - 2n) = 12n - n^2;$$

and from this quadratic is found $n = \pm 3 + 6 = 9$ or 3;

and hence $z = 11 - 8 \times 2$ or $11 - 2 \times 2 = -5$ or 7.

For $n = 3$, and $z = 7$, the series is 11, 9, 7, for which $s = 2$ and for $n = 9$, and $z = -5$, the series is 11, 9, 7, 5, 3, 1, $-$ -3, -5.

4. The first term of an equidifferent series is 1, the last 2, an the number of terms 10: find the common difference and the su of the series.

Here $a = 1$, $z = 2$, $n = 10$, and by [1],

$(n - 1)d = z - a$, or $9d = 2 - 1 = 1$, and $d = \frac{1}{9}$; and by [2]

$$s = \frac{n}{2}(a + z) = \frac{10}{2}(1 + 2) = 5 \times 3 = 15$$

EXERCISES.

1. Find the sum of the natural series of numbers 1, 2, 3, 4, ... arried to 1000 terms, ... = 500500.

2. Required the last term and the sum of the series of odd umbers 1, 3, 5, 7, ... continued to 101 terms,

Last term = 201, and sum = 10201.

3. Find the nth term and the sum of n terms of the natural eries of numbers 1, 2, 3, 4, ...,

The nth term $= n$, and sum of n terms $= \frac{1}{2}n(n+1)$.

4. Find the nth term and the sum of n terms of the series of dd numbers, 1, 3, 5, 7, ...,

The nth term $= 2n - 1$, and the sum of n terms $= n^2$.

5. Find the last term and the sum of 50 terms of the series 2, 4, 6, 8, ..., . . . Last term = 100, and sum = 2550.

6. The first and last terms of an equidifferent progression are 2 and 29, and the common difference is 3: find the number of terms and the sum of the series,

Number of terms = 10, and sum = 155.

7. The first and last terms of a decreasing equidifferent series are 10 and 6, and the number of terms 9: required the common difference and the sum of the series,

Common difference $= \frac{1}{2}$, and sum = 72.

8. The first term of a decreasing equidifferent series is 10, the number of terms 10, and the sum of the series 85: required the last term and the common difference,

Last term = 7, and common difference $= \frac{1}{3}$.

9. The last term of an increasing equidifferent series is 52, the common difference 5, and the sum of the series 297: required the first term and the number of terms,

First term = 2, and number of terms = 11.

10. How many times does the hammer of a common clock strike in a week? . . . . = 1092.

11. A carter has to gravel an avenue with 11 cart-loads of gravel, to be laid down 6 yards distant from each other; the first load to be laid down at the end of the avenue next to a gravel pit, and at the distance of 80 yards from the pit: required the number of yards that he must travel over, supposing that he sets out from the pit, and returns to it after laying down the last load, ... = 2420 yards.

12. Insert four equidifferent means between the terms 5 and 7 of an equidifferent series, ... $= 5\frac{2}{5}, 5\frac{4}{5}, 6\frac{1}{5}$, and $6\frac{3}{5}$.

13. Find the expression for the common difference, order to insert n equidifferent means between the numbers and b, ... Common difference $= d = \frac{a \sim}{n +}$

14. Two travellers (A and B) set out from the same place the same time; A travels at the uniform rate of 3 miles an hou but B's rate of travelling is 4 miles the first hour, $3\frac{1}{2}$ the secon 3 the third, and so on in the same series: in how many hours w A overtake B? . . . . . . Time = 5 hou

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EQUIRATIONAL PROGRESSION.

419. An *equirational progression* is a series, of which the ter increase or decrease in a constant ratio.

The ratio of any term to the preceding term is called t *common ratio.*

Thus, 2, 4, 8, 16, ... is an increasing equirational series, t common ratio of which is 2; and 81, 27, 9, ... is a decreasing or the common ratio being $\frac{1}{3}$. Likewise, a, ar, ar^2, ar^3, ... is a simil series, having the common ratio r; and the series is increasing decreasing according as r is greater or less than 1.

THEOREMS.

420. I. The terms of an equirational progression are in co tinued proportion.

Let a be the first term, and r the ratio, then the series is a, ar^2, ar^3, ... and

$$\frac{ar}{a} = \frac{ar^2}{ar} = \frac{ar^3}{ar^2}, \text{ \&c. or } = r;$$

hence (319) $a : ar = ar : ar^2 = ar^2 : ar^3$, &c.

421. II. The last term of an equirational progression is equ to the product of the first term by that power of the comm ratio whose exponent is one less than the number of terms.

For, the exponent of r in any term is evidently one less th the number expressing the place of the term; thus, in the thi term its exponent is 2, in the fourth it is 3, and so on, and in t nth it is $n - 1$; hence if z be the last term,

$$z = ar^{n-1} \quad \ldots \quad [1].$$

422. III. The product of the first and last terms is equal to that of any two equally distant from the first and last.

For, taking the third term, and the last but two, their product is $= ar^2 \times ar^{n-3} = a^2r^{n-1}$; and that of the first and last is $= a \times ar^{n-1} = a^2r^{n-1}$.

423. Cor. When the number of terms is uneven, the product of the first and last is equal to the square of the middle term; and any term is a mean proportional between the preceding and succeeding terms.

424. The sum of the terms of an equirational series is found by subtracting the first term from the product of the last term and the common ratio, and dividing the difference by one less than the common ratio.

Let the sum of the series

$$a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} = s,$$

then $$ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n = rs,$$

by multiplying the preceding equation by r. Subtracting the former series from the latter,

$$rs - s = ar^n - a, \text{ or } s(r-1) = a(r^n - 1) \text{ or } = rz - a$$

for $ar^n = r \,.\, ar^{n-1} = rz$.

Hence $s = \dfrac{a(r^n - 1)}{r - 1}$, or $s = \dfrac{rz - a}{r - 1}$ ... [2].

When any three of the quantities a, z, r, and s, are given, the fourth may be found from equation [2]. There may be ten cases formed, in which three of the five quantities a, z, r, n, and s, are given, to find the other two from [1] and [2], as in the case of equidifferent series. Some of these cases, however, involve the extraction of high roots, and the use of logarithms, and higher equations than have been treated of in the preceding pages; the formulas for the solution of these cases are all contained in the following table:—

CASES.	GIVEN.	SOUGHT.	FORMULAS.
1.	$r, n, z,$	$a,$	$\frac{z}{r^{n-1}}.$
2.	$r, z, s,$	...	$rz - (r-1)s.$
3.	$n, z, s,$	...	$a(s-a)^{n-1} = z(s-z)^{n-1}.$
4.	$r, n, s,$	...	$s\left(\frac{r-1}{r^n-1}\right).$
5.	$a, r, n,$	$z,$	$ar^{n-1}.$
6.	$a, n, s,$	...	$z(s-z)^{n-1} = a(s-a)^{n-1}.$
7.	$a, r, s,$	...	$s - \frac{s-a}{r}.$
8.	$r, n, s,$	...	$sr^{n-1}\left(\frac{r-1}{r^n-1}\right).$
9.	$a, n, z,$	$r,$	$\left(\frac{z}{a}\right)^{\frac{1}{n-1}}.$
10.	$a, n, s,$	...	$ar^n - sr + s - a = 0.$
11.	$a, z, s,$	...	$\frac{s-a}{s-z}.$
12.	$n, z, s,$	...	$(s-z)r^n - sr^{n-1} + z = 0.$
13.	$a, r, z,$	$n,$	$1 + \frac{\log. z - \log. a}{\log. r}.$
14.	$a, z, s,$	...	$1 + \frac{\log. z - \log. a}{\log. (s-a) - \log. (s-z)}.$
15.	$a, r, s,$	...	$\frac{\log. \{s(r-1)+a\} - \log. a}{\log. r}.$
16.	$r, z, s,$	...	$1 + \frac{\log. z - \log. \{r(z-s)+s\}}{\log. r}.$
17.	$a, n, z,$	$s,$	$(z^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}) \div (z^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}).$
18.	$a, r, n,$	...	$a\left(\frac{r^n-1}{r-1}\right).$
19.	$a, r, z,$	...	$\frac{rz-a}{r-1}.$
20.	$r, n, z,$	...	$\frac{z}{r^{n-1}}\left(\frac{r^n-1}{r-1}\right).$

EXAMPLES.

1. Find the ninth term of the series 1, 3, 9, 27, ... and the sum of the first nine terms.

Here $a = 1$, $r = 3$, and $n = 9$;

hence $$z = ar^{n-1} = 1 \times 3^{9-1} = 3^8 = 6561,$$

and $$s = \frac{rz - a}{r - 1} = \frac{3 \times 6561 - 1}{3 - 1} = \frac{19683 - 1}{2} = \frac{19682}{2} = 9841.$$

2. The first term of a decreasing equirational series is 1, the common ratio $\frac{1}{3}$, and the number of terms 5: required the last term and the sum of the series.

Here $a = 1$, $r = \frac{1}{3}$, and $n = 5$;

hence $$z = ar^{n-1} = 1 \times \left(\frac{1}{3}\right)^{5-1} = \left(\frac{1}{3}\right)^4 = \frac{1}{81},$$

and $$s = \frac{rz - a}{r - 1} \text{ or } = \frac{a - rz}{1 - r} = \frac{1 - \frac{1}{3} \times \frac{1}{81}}{1 - \frac{1}{3}} = \frac{1 - \frac{1}{243}}{\frac{2}{3}} = \frac{242}{243} \times \frac{3}{2}$$

$$= \frac{121}{81} = 1\frac{40}{81}.$$

3. Insert three equirational means between the numbers 3 and 48.

This question is the same as if the first and last term, and the number of terms of an equirational series, were given, to find the common ratio.

By [1], $z = ar^{n-1}$; $\therefore\ r^{n-1} = \frac{z}{a}$,

and in this example $a = 3$, $z = 48$, and $n = 5$;

hence $r^{5-1} = \frac{z}{a}$ or $r^4 = \frac{48}{3} = 16$, and $r^2 = \sqrt{16} = 4$, and consequently

$$r = \sqrt{4} = 2.$$

The means are therefore 6, 12, and 24.

EXERCISES.

1. The first term of an equirational series is 3, the common ratio 2, and the number of terms 10: required the last term and the sum of the series, ... = 1536 and 3069.

2. A person walks 4 miles the first hour, 2 the second, 1 the third, and so on, in equirational progression, and continues his

journey for 10 hours: how far does he travel the last hour, and what distance does he travel altogether?

$$\text{Last hour} = z = \frac{1}{128}, \text{ and entire distance} = s = 7\frac{127}{128}.$$

3. Find the sum of the series 1, 3, 9, 27, ... continued to 12 terms, ... Sum = 265720.

4. In order to insert seven equirational means between the numbers 16 and $\frac{1}{16}$, what must be the common ratio?
Common ratio = $\frac{1}{2}$.

5. The first term of an equirational series is 3, the last term 192, and the number of terms 7: find the common ratio and the sum of the series, . . . Common ratio $= r = 2$, $s = 381$.

6. Insert 4 equirational means between 5 and 160, and find the sum of the 6 terms,
Equirational means = 10, 20, 40, and 80, and $s = 315$.

425. In finding the sum of a decreasing equirational series carried out to an indefinite extent, or, in other words, an infinite decreasing equirational series, the last term may be considered as vanishing, or becoming = 0.

Hence, since $z = 0$ the sum of such a series is

$$s = \frac{-a}{r-1}, \text{ or } s = \frac{a}{1-r};$$

or, since in this case r is a proper fraction, if it be represented by $\frac{p}{q}$, where $p \angle q$, then

$$s = \frac{a}{1 - \frac{p}{q}}, \text{ or } s = \frac{aq}{q-p}.$$

EXAMPLES.

1. Find the sum of the infinite series 1, $\frac{1}{2}$, $\frac{1}{4}$, ...

Here $a = 1$, and $\frac{p}{q} = \frac{1}{2}$, or $p = 1$, $q = 2$;

$$\therefore s = \frac{aq}{q-p} = \frac{2}{2-1} = 2.$$

2. Find the sum of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, ... carried to infinity.

Here $a = 1$, $p = 2$, $q = 3$;

$$\therefore s = \frac{aq}{q-p} = \frac{1 \times 3}{3-2} = 3.$$

3. Find the sum of the infinite series $1, -\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \ldots$

Here $a = 1$, $r = -\frac{1}{4}$, or $\frac{p}{q} = -\frac{1}{4}$, $p = -1$, $q = 4$;

$$\therefore s = \frac{aq}{q-p} = \frac{1 \times 4}{4+1} = \frac{4}{5}.$$

EXERCISES.

1. Find the sum of the series $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ to infinity, $= \frac{3}{2}$.

2. ... ... $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$ to infinity, $= \frac{3}{4}$.

3. If a and x be each greater than unity, find the sum of the series $a, \frac{1}{x}, \frac{1}{ax^2} \ldots$ to infinity, ... $= \frac{a^2x}{ax-1}$.

426. Recurring decimals—that is, repeaters and circulates—may be represented in the form of infinite decreasing equirational series, and their values found in the form of vulgar fractions by summing the series; whence the rules for converting them into vulgar fractions can be derived.

EXAMPLES.

1. Find the value of the repeater $\cdot\dot{4}$.

$$\cdot\dot{4} = \cdot 444 \ldots = \frac{4}{10} + \frac{4}{100} + \ldots$$

Here $a = \frac{4}{10}$, and $r = \frac{1}{10}$,

and $$\therefore s = \frac{a}{1-r} = \frac{\frac{4}{10}}{1 - \frac{1}{10}} = \frac{4}{10} \times \frac{10}{9} = \frac{4}{9}.$$

2. If n be the repeating digit, and the value of the fraction s,

then $$s = \frac{n}{10} + \frac{n}{100} + \frac{n}{1000} + \ldots$$
$$= \frac{n}{10} + \frac{n}{10^2} + \frac{n}{10^3} + \ldots;$$

hence $$\therefore s = \frac{a}{1-r} = \frac{\frac{n}{10}}{1 - \frac{1}{10}} = \frac{n}{10} \times \frac{10}{9} = \frac{n}{9}.$$

From this result the common rule for reducing a pure repeater to a vulgar fraction is derived.

3. Find the value of the circulate $\cdot\dot{2}\dot{4}$.

$$\cdot\dot{2}\dot{4} = \cdot 242424 \ldots = \frac{24}{100} + \frac{24}{10000} + \frac{24}{1000000} + \ldots$$

Here $a = \dfrac{24}{100}$, and $r = \dfrac{1}{100}$;

$$\text{hence } \therefore s = \frac{a}{1-r} = \frac{\dfrac{24}{100}}{1 - \dfrac{1}{100}} = \frac{24}{100} \times \frac{100}{99} = \frac{24}{99}.$$

4. Find the value of the mixed repeater $\cdot 4\dot{3}$.

$$\cdot 4\dot{3} = \cdot 4 + \cdot 0\dot{3}, \text{ and } \cdot 0\dot{3} = \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots$$

Hence for the value of $\cdot 0\dot{3}$.

$$s = \frac{a}{1-r} = \frac{\dfrac{3}{100}}{1 - \dfrac{1}{10}} = \frac{3}{100} \times \frac{10}{9} = \frac{3}{90},$$

and therefore

$$\cdot 4\dot{3} = \frac{4}{10} + \frac{3}{90} = \frac{9 \times 4 + 3}{90} = \frac{10 \times 4 + 3 - 4}{90} = \frac{43 - 4}{90} = \frac{39}{90}.$$

Or thus: let $\cdot 4\dot{3}$ or $\cdot 43333 \ldots = s$,

then $4\cdot 333 \ldots = 10s$,

and $43\cdot 333 \ldots = 100s$,

subtracting $43 - 4 = 90s$,

or $s = \dfrac{43 - 4}{90} = \dfrac{39}{90}$.

5. Find the value of the mixed circulate $\cdot 4\dot{3}\dot{5}$.

$$\cdot 4\dot{3}\dot{5} = \cdot 4 + \cdot 0\dot{3}\dot{5}, \text{ and } \cdot 0\dot{3}\dot{5} = \frac{35}{1000} + \frac{35}{100000} + \ldots$$

and the value of

$$\cdot 0\dot{3}\dot{5} \text{ is } s = \frac{a}{1-r} = \frac{\dfrac{35}{1000}}{1 - \dfrac{1}{100}} = \frac{35}{1000} \times \frac{100}{99} = \frac{35}{990};$$

and therefore

$$\cdot 4\dot{3}\dot{5} = \frac{4}{10} + \frac{35}{990} = \frac{4 \times 99 + 35}{990} = \frac{4 \times 100 + 35 - 4}{990}$$

$$= \frac{435 - 4}{990} = \frac{431}{990};$$

or thus: let $\cdot 4\dot{3}\dot{5}$ or $\cdot 4353535 \ \ldots = s$,

then $4 \cdot 353535 \ \ldots = 10s$,

and $435 \cdot 353535 \ldots = 1000s$,

subtracting $435 - 4 = 990s$,

or $s = \dfrac{435 - 4}{990} = \dfrac{431}{990}$.

EXERCISES.

Find, by summing the series, the value of the pure repeater $\cdot\dot{5}$; the pure circulate $\cdot\dot{3}\dot{6}$; the mixed repeater $\cdot 4\dot{6}$; and of the mixed circulate $\cdot 53\dot{2}4\dot{1}$. The values are respectively $= \frac{5}{9}, \frac{4}{11}, \frac{7}{15}$, and $\frac{13297}{24975}$.

HARMONIC PROGRESSION.

DEFINITIONS.

427. Three quantities are said to be in *harmonic progression* when the first is to the third as the difference between the first and second is to the difference between the second and third; and the three quantities are called *harmonic progressionals.*

Thus, if a, b, c, be in harmonic progression, then $a : c = a - b : b - c$.

428. When three quantities are in harmonic progression, the second is said to be an *harmonic mean* between the other two; and the third is called a *third harmonic progressional* to the first and second.

429. Any series of quantities are said to be in *harmonic progression*, or to be *harmonic progressionals*, when every consecutive three are in harmonic progression.

Thus, a, b, c, d, e, ... are in harmonic progression if a, b, c, and b, c, d, and c, d, e, ... be in harmonic progression.

PROBLEMS AND THEOREMS.

430. I. To find a *harmonic mean* between two quantities.

Let a, b, be the two quantities, and x the harmonic mean, the (427) $a : b = a - x : x - b$, or $a(x - b) = b(a - x)$;

hence therefore $$x = \frac{2ab}{a + b}.$$

431. II. To find a third harmonic progressional to any tw quantities.

Let a, b, be the quantities, and x the third harmonic progres sional, then (427)

$$a : x = a - b : b - x, \text{ or } a(b - x) = x(a - b);$$

and from this equation, $x = \dfrac{ab}{2a - b}$.

432. III. Hence if any two of three harmonic progressional be given, the third can be found.

433. IV. If three quantities be in harmonic progression, thei reciprocals are in equidifferent progression.

Let a, b, and c, be in harmonic progression, then $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ are in equidifferent progression.

For $a : c = a - b : b - c$, or $ab - ac = ac - bc$,

or dividing by $a\ b\ c$, $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$, or $\frac{1}{c}, \frac{1}{b}, \frac{1}{a}$ are in equidifferen progression (410); in the same manner it may be proved tha if b, c, d be in harmonic progression, $\frac{1}{d} - \frac{1}{c} = \frac{1}{c} - \frac{1}{b}$; an therefore, $\frac{1}{d}, \frac{1}{c}, \frac{1}{b}$ are in equidifferent progression. The same metho of proof may be extended to any number of terms.

COR. By this article any number of harmonic means may b inserted between two extremes, by inserting the same number equidifferent means between the reciprocals of the extremes, an their reciprocals will be the harmonic means required.

434. V. To insert two harmonic means between two numbers.

Let a, b, be the two numbers, and x, y, two harmonic means between them,

then $$a : y = a - x : x - y,$$

and $$x : b = x - y : y - b,$$

from which are found the two equations

$$a(x - y) = (a - x)y, \text{ and } x(y - b) = b(x - y),$$

or $$ax + xy - 2ay = 0, \text{ and } by + xy - 2bx = 0.$$

A value of y being found from each of these equations, and these values being equated, give

$$\frac{ax}{2a - x} = \frac{2bx}{b + x}, \text{ or } \frac{a}{2a - x} = \frac{2b}{b + x},$$

or $$ab + ax = 4ab - 2bx;$$

hence $$x = \frac{3ab}{a + 2b},$$

and substituting this value of x in either of the values of y, the value of y is found, or

$$y = \frac{3ab}{2a + b}.$$

435. In a similar manner, any number of harmonic means may be inserted between two numbers, for as many quadratic equations to determine them could be formed.

EXAMPLES.

1. Find a harmonic mean between 1 and $\frac{2}{3}$.

Here $a = 1$, $b = \frac{2}{3}$, and

by (430), $x = \dfrac{2ab}{a + b} = \dfrac{2 \times 1 \times \frac{2}{3}}{1 + \frac{2}{3}} = \dfrac{4}{3} \times \dfrac{3}{5} = \dfrac{4}{5}$ the harmonic mean.

The result is verified by the proportion $1 : \dfrac{2}{3} = 1 - \dfrac{4}{5} : \dfrac{4}{5} - \dfrac{2}{3}$,

or $\dfrac{3}{3} : \dfrac{2}{3} = \dfrac{1}{5} : \dfrac{2}{15}$, or $\dfrac{3}{15} : \dfrac{2}{15}$, or $3 : 2 = 3 : 2$.

Or it is verified thus by article (433) $\dfrac{3}{2} - \dfrac{5}{4} = \dfrac{5}{4} - 1$; that is,

$\dfrac{5}{4}$ is an equidifferent mean between 1 and $\dfrac{3}{2}$.

2. Find a third harmonic progressional to 6 and 4.

Here $a = 6$, $b = 4$, and

by (431), $\therefore x = \dfrac{ab}{2a - b} = \dfrac{6 \times 4}{12 - 4} = \dfrac{24}{8} = 3.$

3. Insert two harmonic means between 1 and $\frac{2}{3}$.

By (434), $a = 1$, $b = \frac{2}{3}$, and

$$\therefore x = \frac{3ab}{a + 2b} = \frac{3 \times 1 \times \frac{2}{3}}{1 + 2 \times \frac{2}{3}} = 2 \times \frac{3}{7} = \frac{6}{7},$$

and $$y = \frac{3ab}{2a + b} = \frac{3 \times 1 \times \frac{2}{3}}{2 + \frac{2}{3}} = 2 \times \frac{3}{8} = \frac{3}{4}.$$

The harmonic series therefore is $1, \frac{6}{7}, \frac{3}{4}$, and $\frac{2}{3}$.

EXERCISES.

1. Find a harmonic mean between 12 and 6, . . $= 8$.

2. Find a third harmonic progressional to 3 and 2, . $= \frac{3}{2}$.

3. Find two harmonic means between 84 and 56, $= 72$ and 63.

4. Theorem I. If four quantities be in harmonic progression, the product of the first and second is to that of the third and fourth, as the difference between the two former to the difference between the two latter.

5. Theorem II. In any harmonic series, the product of the first two terms is to that of the last two as the difference between the two former to that between the two latter.

PROPERTIES OF NUMBERS.

Several properties of numbers were given in a former section onnected with the least common multiple and greatest common neasure of quantities; in this section a few more properties are dded, which are interesting and useful both in a practical and heoretical point of view.

THEOREMS.

436. I. If x and y represent the two digits respectively in the lace of tens and of units, composing a number expressed by the otation of the common or decimal system of numeration, the umber would be expressed algebraically; thus—

$$10x + y.$$

If the number consisted of three digits, x, y, z, the last being that in the place of units, it would be expressed thus—

$$100x + 10y + z, \text{ or } 10^2x + 10y + z.$$

For any digit has only its real value in the place of units, but when it is in the place of tens, its nominal value is 10 times greater, 10 being the base of the system; in the place of hundreds, it is 100 times greater; and so on; its nominal value increasing 10 times for every place it is removed to the left. Thus, 6542

$= 6000 + 500 + 40 + 2 = (6 \times 1000) + (5 \times 100) + (4 \times 10) + 2.$

If $a = 2$, $b = 4$, $c = 5$, $d = 6$, the number would be expressed by

$$1000d + 100c + 10b + a, \text{ or } a + 10b + 10^2c + 10^3d.$$

437. II. If a number be divided by 9, the remainder is the same as that resulting from dividing the sum of its digits by 9.

Let the number be $a + 10b + 100c + 1000d + \dots = N$; then dividing this number N by 9, the remainders are a, b, c, d, ...; hence the remainder, when N is divided by 9, is the remainder when $a + b + c + d + \dots$ is divided by 9.

438. Cor. Hence when the sum of the digits of a number is divisible by 9, so is the number itself.

The same property belongs to the number 3.

439. III. The remainder, on dividing a number by 6, is the same as that resulting from dividing by 6 the sum of the digit in the place of units added to 4 times the sum of its other digits.

For the number $a + 10b + 100c + 1000d + \ldots$ being divided by 6, gives for a remainder

$$a + 4b + 4c + 4d + \ldots = a + 4(b + c + d + \ldots),$$

when this latter quantity therefore is divisible by 6, so is the number itself.

440. IV. If two numbers be divided by a third, and the product of their remainders be also divided by the same number, the remainder arising from this last division is the same as that resulting from the division of the product of the two given numbers by the same number.

Let the given numbers be N and N', and D the divisor, and let q and q' be the quotients, and r and r' the remainders;

then $$N = qD + r, \text{ and } N' = q'D + r';$$

hence $$NN' = qq'D^2 + (qr' + q'r)D + rr';$$

and since the first two terms of the second member are divisible by D, the remainder will arise merely from the division of rr' by D, which is therefore the remainder when the product NN' is divided by D.

This property is useful in testing the accuracy of the product of two numbers. Any number may be taken as the divisor D; but as the remainder, when a number is divided by 9, is so easily obtained (437), being the same as that arising from dividing the sum of the digits by 9, or, as it is called, by *throwing out the nines*, this number is therefore commonly adopted for proving the accuracy of multiplication.

441. V. Let r be the base of any system of numeration, and $a, b, c, \ldots$ the digits of any number taken in order, beginning at the place of units, then the number is expressed by

$$a + br + cr^2 + dr^3 + \ldots$$

If $r = 10$, the number would be expressed in the common or decimal system. If $r = 8$, the system is the octary, and the value of a figure would increase 8 times for every place it is removed from the units' place, and the above number would be

$$a + 8b + 8^2c + 8^3d + \ldots, \text{ or } a + 8b + 64c + 512d + \ldots,$$

but in this scale 8 would be denoted by 10, the 1 being in the place of eights.

442. VI. In any system of numeration, the difference between a number and the sum of its digits is divisible by a number one less than the base of the scale.

Let r be the base, and the number

$$= a + br + cr^2 - dr^3 + \ldots \text{ then } a + b + c + d + \ldots$$

s the sum of the digits, and the difference between these uantities

$$= b(r-1) + c(r^2-1) + d(r^3-1) + \ldots,$$

vhich is evidently divisible by $r - 1$.

Thus, in the decimal system $r = 10$, and if 6432 be a number, hen $6432 - (6 + 4 + 3 + 2) = 6432 - 15 = 6417$, which is ivisible by $9 = r - 1$.

The place of units is reckoned the *first* place.

443. VII. If the excess of the sum of the digits in the odd laces of a number above the sum of those in the even places be ivisible by a number a unit greater than the base, the given umber itself is divisible by the same number.

Let the number be $a + br + cr^2 + dr^3 + \ldots = N$,

nd let $\qquad a + c + e + \ldots = P$,

nd $\qquad b + d + f + \ldots = Q$,

hen $\quad N = P - Q + b(r+1) + c(r^2-1) + d(r^3+1) + \ldots,$

nd the terms after $P - Q$ in the second member are evidently ivisible by $(r + 1)$; and therefore if $P - Q$ be divisible by $r + 1$, he number N is so also.

Thus, on the decimal scale, $r + 1 = 11$, and if the number is 36958476, $P = 21$, $Q = 32$, and $P - Q = 21 - 32 = -11$, vhich is divisible by 11, and the number itself, therefore, is also ivisible by 11.

444. Cor. Hence $N - P + Q$ is divisible by $r + 1$.

EXAMPLES.

1. In what system has the number 32 the same value as 14 in he decimal system?

Let $x =$ the base of the numerical system,

hen $\qquad 32 = 3x + 2 = 10 + 4 = 14,$

$$3x = 14 - 2 = 12,$$

$$\therefore x = 4.$$

Hence the system is the quaternary.

In this system $32 = 3 \times 4 + 2 = 12 + 2 = 14$.

2. In what system is the number 2310 equal to 45 times the base?

Let $x =$ the base,

then $2310 = 2x^3 + 3x^2 + x + 0 = 45x$,

or $2x^2 + 3x = 45 - 1 = 44$.

From this equation is found $x = 4$.

3. In the binary system $r = 2$, and the number of digits is two—namely, 0 and 1—and the greatest number consisting of 4 places i 1111, which is $= 1 \times 2^3 + 1 \times 2^2 + 1 \times 2 + 1 = 1 \times 8 + 1 \times$ $+ 1 \times 2 + 1 = 8 + 4 + 2 + 1 = 15$. Hence all numbers, fror 1 to 15 inclusive, must be capable of being expressed in thi system by 1 repeated not more than 4 times. But all the possibl values of 1 in these 4 places are 1, 2, 4, or 8; hence all number from 1 to 15 inclusive, are made up of some of the combination of 1, 2, 4, and 8.

Hence four weights of 1, 2, 4, and 8 ounces, will be sufficient t make up any number of ounces from 1 to 15 inclusive.

This curious proposition may be easily extended.

EXERCISES.

1. If the first digit in any number be divisible by 2, so also i the number.

2. Any number having 5 or 0 in its place of units, is divisibl by 5.

3. If a number be divided by 3, the remainder will be the sam as if the sum of its digits were divided by 3.

4. A number is divisible by 4 when the number composed of it first two digits is so; or when the sum of the first digit and twic the second is so.

5. A number is divisible by 8 when the number composed c its first three digits is so; or when the sum of the first digit, twic the second, and four times the third, is so.

6. If the sum of the odd digits and r times the even digits of number (r being the base) be divisible by $r + 1$, or $r - 1$, c $r^2 - 1$, the number itself is so.

7. If the sum of the even digits of a number, expressed in th decimal system, be added to the number itself, and the sum c the odd digits be subtracted from it, the resulting number will b divisible by 11.

8. In what system is the value of the number 231 equal to tha of 66 in the decimal system?

9. The square of every even number is divisible by 4 [illegible] square of every odd number diminished by 1 is also divis[illegible]

10. The difference of the squares of two odd numbers, as well as of two even numbers, is divisible by 8.

11. The product of any two odd or two even numbers is less than the square of half their sum by the square of half their difference.

PERMUTATIONS.

445. The various orders in which objects are capable of being arranged in succession are called their *permutations*.

Thus, the letters a, b, c, when taken in pairs, will form six permutations ab, ba, ac, ca, bc, cb; and when the three are taken, they will also form six—namely, abc, acb, bac, bca, cab, cba.

THEOREMS.

446. I. The number of permutations that can be formed with n letters, taken two and two, is $= n(n-1)$.

Let a, b, c, ... be n letters, then a may be placed before each of the remaining letters, which are $n-1$ in number, and thus $n-1$ permutations are formed, in which a stands first. So b may be placed before each of the other letters; and thus $n-1$ permutations are formed, in which b stands first. The same may be said of all the letters which are n in number. Hence there are $n-1$ permutations, repeated n times, or altogether, there are $n(n-1)$ permutations.

447. II. The number of permutations that can be formed with n letters, taken three and three, is $n(n-1) \times (n-2)$.

For any one of the permutations, taken two and two, can be placed before each of the remaining letters, which are $n-2$ in number; and thus $n-2$ permutations, taken three and three, are formed. Thus, if a, b, c, d, e, ... be the n letters, then the permutation ab may be placed before each of the letters c, d, e, ... which are $n-2$ in number; and thus are formed $n-2$ permutations, taken three and three—namely, abc, abd, abe, ... So the permutation ac being placed before each of the letters b, d, e, ... will form $n-2$ permutations, taken three and three. The same may be proved of all the other permutations, taken two and two; and thus all the possible permutations, taken three and three, will be formed. It might be said that the first permutation, taken alone—namely, ab—may not only be placed before c forming the permutation abc, but that it might also be placed after c, so as

to form another permutation cab; but this last permutation : otherwise formed—namely, by placing the permutation ca befo: b; so that to form *all* the permutations, taken three and three, is merely necessary to place *each* of the permutations, taken tw and two, *before* each of the remaining $n - 2$ letters. Hence th whole number of permutations, taken three and three, will b equal to the number of them taken two and two repeated $n -$ times, or $n(n - 1) \times (n - 2)$.

448. III. It may be similarly proved, that the number of pe mutations of n letters, taken four and four together, is $n(n - 1$ $(n - 2)(n - 3)$, by placing each of the permutations, taken thre and three, before each of the remaining letters, which are $n -$ in number.

By proceeding in this manner, the following general propositio is arrived at by induction; namely—

449. IV. The number of permutations of n letters, taken r an r together, is $= n(n - 1)(n - 2) \ldots (n - r + 1)$; and hence—

The number of permutations of n letters, when all of them ar taken, is $n(n - 1)(n - 2) \ldots 3 \times 2 \times 1$, or $1 \times 2 \times 3 \times$ .. $(n - 2)(n - 1)n$.

For in this case $r = n$, and $n - r + 1 = n - n + 1 = 1$; th preceding factor is $n - n + 2 = 2$; the one preceding this last i $= n - n + 3 = 3$; and so on.

450. V. The number of permutations of n letters, taken $n -$ and $n - 1$ together, is the same as when they are all taken; o $= 2 \times 3 \times 4 \times \ldots (n - 2)(n - 1)n$.

For in this case $r = n - 1$, and $r - 1 = n - 2$; and henc $n - r + 1 = n - n + 2 = 2$, $n - r + 2 = n - n + 3 = 3$; and so o

EXAMPLES.

1. In how many different orders can five persons sit on a form

The number of permutations of 5 objects taken altogether i $= 1 \times 2 \times \ldots (n - 1)n = 1 \times 2 \times 3 \times 4 \times 5 = 120$.

2. How many signals may be made with 4 flags?

Number when taken
singly is . . . . = 4
2 by 2 is $n(n - 1) = 4 \times 3$, ... = 12
3 by 3 is $n(n - 1)(n - 2) = 4 \times 3 \times 2$, . . . = 24
4 by 4 is $n(n - 1)(n - 2)(n - 3) = 4 \times 3 \times 2 \times 1$, = 24

∴ the total number is . . . . = 64

EXERCISES.

1. How many permutations of four notes each, sounded successively, can be formed with the seven musical notes of one octave? . . . . = 840.

2. In how many different ways can the seven prismatic colours be arranged? . . . . = 5040.

3. In how many different ways can six letters be arranged when taken singly, two by two, three by three, and so on, till they are all taken together? . . . . = 1956.

COMBINATIONS.

451. The different collocations that can be formed by any number of objects, without regarding the order in which they are arranged, are called their *combinations*.

Although several permutations may consist of the same objects, every two combinations must consist of different ones.

THEOREMS.

452. I. The number of combinations of n objects, taken two and two, is $= \dfrac{n(n-1)}{1 \times 2}$.

For the number of permutations, two and two is $= n(n-1)$; and for each combination there are two permutations; as, for example, the combination ab affords two permutations ab, ba; therefore the number of combinations is $\frac{1}{2}$ of the permutations, or they are $= \dfrac{n(n-1)}{1 \times 2}$.

453. II. The number of combinations of n objects, taken three and three, is $= \dfrac{n(n-1)(n-2)}{1 \times 2 \times 3}$.

For the number of permutations, taken three and three, is $n(n-1)(n-2)$, and each combination, as abc, affords 6, or 2×3 permutations; hence the number of combinations

$$= \frac{n(n-1)(n-2)}{1 \times 2 \times 3}.$$

454. III. Generally, the number of combinations of n objects, taken m and m together, is

$$= \frac{n(n-1)(n-2)\ldots(n-m+1)}{1 \times 2 \times 3 \ldots m}.$$

For the number of permutations, taken m and m together, is $= n(n-1)(n-2)\ldots(n-m+1)$, and each combination of m objects affords $1 \times 2 \times 3 \ldots.(m-1)m$ permutations; hence the number of combinations is

$$= \frac{n(n-1)(n-2)\ldots(n-m+1)}{1 \times 2 \times 3 \times \ldots m}.$$

455. IV. When all the quantities are taken together, there is evidently only one combination—as appears also from the consideration that m is then $= n$, and the last formula becomes

$$\frac{n(n-1)(n-2)\ldots.3 \times 2 \times 1}{1 \times 2 \times 3 \ldots n} = 1.$$

EXAMPLE.

How many products can be formed with 5 different quantities?

The number when taken

2 and 2 is $= \frac{n(n-1)}{2} = \frac{5 \times 4}{2}$, . . . $= 10$

3 and 3 is $= \frac{n(n-1)(n-2)}{2 \times 3} = \frac{5 \times 4 \times 3}{2 \times 3}$, . . $= 10$

4 and 4 is $= \frac{n(n-1)(n-2)(n-3)}{2 \times 3 \times 4} = \frac{5 \times 4 \times 3 \times 2}{2 \times 3 \times 4}$, $= 5$

all together, ... $= 1$

Total number, . . . $= 26$

EXERCISES.

1. How many different tints of colour can be formed by mixing the seven prismatic colours always in the same proportion, and in every possible way? . . . . $= 120$

2. If five of eight quantities are always given to find the other three, how many cases may be formed in reference to the data, and how many in reference to the quantities required? $= 56$ and 168

3. How many different notes may be rung on ten different bells supposing all the combinations to produce different notes? $= 1023$

4. How many different weights, consisting of a whole number of pounds, can be formed by means of the six weights 1, 2, 4, 8, 16, and 32 pounds, taking them singly, and then combining them two by two, three by three, and so on, till they are all taken together? . . . . $= 63$.

456. It appears from this result that, as 63 is the greatest weight, or that obtained by taking all the weights together, and as there are also 63 combinations, all of which are whole numbers of pounds, these six weights, variously combined, form a series of weights represented by the natural series of numbers, 1, 2, 3, 4, ... up to 63 (see ART. 444, EXAMP. 3.)

It might be shewn generally that any number of weights represented by $1, 2, 2^2, 2^3, \ldots 2^n$, would afford a series denoted by the natural series 1, 2, 3, 4, 5, ... up to $2^{n+1} - 1$, which is just the sum of the series $1 + 2 + 2^2 + 2^3 + \ldots + 2^n$; for

$$s = \frac{rl - a}{r - 1} = \frac{2 \times 2^n - 1}{2 - 1} = 2^{n+1} - 1.$$

METHOD OF UNDETERMINED COEFFICIENTS.

457. The method of undetermined coefficients is a method for the development of certain algebraic functions, such as rational fractions having compound denominators, or compound irrational quantities, into infinite series, arranged according to the ascending powers of one of the letters, which is considered to be variable, or to be subject to any arbitrary value.

The function is assumed equal to an infinite series, such as

$$A + Bx + Cx^2 + Dx^3 + \ldots,$$

in which the value of x is unrestricted, but the values of its coefficients are constant, though unknown or undetermined. The values of the coefficients may be determined on the following principle:—

458. Let the value of x in the equation

$$A + Bx + Cx^2 + Dx^3 + \ldots = 0$$

be arbitrary; that is, let the equation be fulfilled by any value whatever of x. An equation of this kind is said to be *identical* (255), to distinguish it from an ordinary equation, which is satisfied only by certain values of the unknown quantity. Such an equation is satisfied by its coefficients, whether the equation be a finite or an infinite series; for the value of x being arbitrary, and those of the

coefficients constant, or always the same, whatever be the value of x, if x be assumed $= 0$, the series is reduced to one term; namely—

$$A = 0;$$

and as the value of A is the same for any value of x, it is always $= 0$; and hence the series becomes

$$Bx + Cx^2 + Dx^3 + \ldots = 0,$$

or

$$B + Cx + Dx^2 + \ldots = 0;$$

and if $x = 0$, then $B = 0$; and in a similar manner it may be proved that $C = 0$, $D = 0$, &c.

459. Hence if two equal polynomials

$$A + Bx + Cx^2 + \ldots = A' + B'x + C'x^2 + \ldots$$

be *identical*—that is, if they merely represent the same quantity in different forms, and be therefore always equal, whatever value be assigned to x—then the coefficients of their *homologous* terms—that is, of the terms containing the same power of x—are equal.

For this identical equation may be put under the form

$$A - A' + (B - B')x + (C - C')x^2 + (D - D')x^3 + \ldots = 0;$$

and hence (458)

$$A - A' = 0,\ B - B' = 0,\ C - C' = 0,\ D - D' = 0,\ \ldots,$$

and therefore

$$A = A',\ B = B',\ C = C',\ D = D',\ \ldots$$

Let $F(x)$ denote the algebraical function to be expanded, and assume

$$F(x) = A + Bx + Cx^2 + Dx^3 + \ldots$$

which constitutes an *identity*; that is, an equality between a quantity not developed and its development. If the function be a fraction, multiply both sides by its denominator; or if it be an irrational quantity, involve both sides to the corresponding power; then the coefficients of the homologous terms being equated, and the coefficients of the other terms assumed $= 0$, the resulting equations will determine the coefficients; or if the first side be transposed, and the coefficients of each of the powers of x be assumed $= 0$, the required coefficients may be determined.

EXAMPLES.

1. Find the development of $\dfrac{a}{b + cx}$.

Let $\dfrac{a}{b + cx} = A + Bx + Cx^2 + Dx^3 + \ldots$

Here the undetermined coefficients A, B, C, ... are functions,

as yet unknown, of the constants a, b, c, of the given function. To find these coefficients, multiply both members by $b + cx$, and transpose a; then

$$\left.\begin{array}{l|l|l|l} Ab + Bb & x + Cb & x^2 + Db & x^3 + \dots \\ -a + Ac & \ +Bc & \ + Cc & \quad \dots \end{array}\right\} = 0.$$

Hence (458) $Ab - a = 0$, $Bb + Ac = 0$, $Cb + Bc = 0$, $Db + Cc = 0, \dots$

from which

$$A = \frac{a}{b},\ B = -\frac{Ac}{b} = -\frac{ac}{b^2},\ C = -\frac{Bc}{b} = \frac{ac^2}{b^3}, \dots$$

and $$\therefore \frac{a}{b + cx} = \frac{a}{b} - \frac{ac}{b^2}x + \frac{ac^2}{b^3}x^2 - \frac{ac^3}{b^4}x^3 + \dots$$

$$= \frac{a}{b}\left(1 - \frac{c}{b}x + \frac{c^2}{b^2}x^2 - \frac{c^3}{b^3}x^3 + \dots\right)$$

2. Convert $\sqrt{(a^2 - x^2)}$ into an infinite series.

If this quantity were assumed equal to the series $A + Bx + Cx^2 + Dx^3 +, \dots$ it would be found in determining the coefficients that those of the odd powers of x, namely, $B, D, F, \dots$ are $= 0$; and therefore the series may be changed into one containing only even powers of x; thus, let

$$\sqrt{(a^2 - x^2)} = A + Bx^2 + Cx^4 + Dx^6, \dots$$

squaring both sides, and transposing $a^2 - x^2$, we find

$$\left.\begin{array}{l|l|l|l} A^2 + 2AB & x^2 + B^2 & x^4 + 2AD & x^6 + \dots \\ -a^2 + \quad 1 & \quad + 2AC & \quad + 2BC & \quad \dots \end{array}\right\} = 0.$$

Hence $A^2 - a^2 = 0$; $2AB + 1 = 0$; $B^2 + 2AC = 0$; $2AD + 2BC = 0; \dots$

from which

$$A = a;\ B = -\frac{1}{2A} = -\frac{1}{2a};\ C = -\frac{B^2}{2A} = -\frac{1}{8a^3},$$

$$D = -\frac{BC}{A} = -\frac{1}{16a^5}; \qquad \dots$$

$$\therefore \sqrt{(a^2 - x^2)} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \dots$$

3. To resolve $\dfrac{5x - 4}{(x + 1)(x - 2)}$ into separate or partial fractions.

Assume $\dfrac{5x-4}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$;

then $$5x - 4 = A(x-2) + B(x+1), \qquad [1].$$

Since this equation is true for all values of x, let $x + 1 = 0$, or $x = -1$; then $-9 = -3A$; $\therefore A = 3$.

Next let $x - 2 = 0$, or $x = 2$;

then $$6 = 3B; \therefore B = 2,$$

and hence $\dfrac{5x-4}{(x+1)(x-2)} = \dfrac{3}{x+1} + \dfrac{2}{x-2}$.

The same result may be obtained from equation (1) by equating the coefficients of x, and the constant terms on each side; thus—

$$A + B = \quad 5,$$

and $$-2A + B = -4,$$

from which we find by the ordinary method $A = 3$ and $B = 2$.

4. To resolve $\dfrac{1}{(x-a)(x-b)(x-c)}$ into partial fractions.

Assume $\dfrac{1}{(x-a)(x-b)(x-c)} = \dfrac{A}{x-a} + \dfrac{B}{x-b} + \dfrac{C}{x-c}$;

$\therefore 1 = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$.

Let $x = a$;

then $1 = A(a-b)(a-c)$, or $A = \dfrac{1}{(a-b)(a-c)}$.

Let $x = b$;

then $1 = B(b-a)(b-c)$; or $B = -\dfrac{1}{(a-b)(b-c)}$.

Let $x = c$;

then $1 = C(c-a)(c-b)$; or $C = \dfrac{1}{(a-c)(b-c)}$;

$$\therefore \frac{1}{(x-a)(x-b)(x-c)} = \frac{1}{(a-b)(a-c)(x-a)}$$
$$- \frac{1}{(a-b)(b-c)(b-x)} + \frac{1}{(a-c)(b-c)(c-x)}.$$

5. To resolve $\dfrac{3x^3 - 10x^2 + 12x - 2}{x(x-1)^3}$ into partial fractions.

Assume

$$\frac{3x^3 - 10x^2 + 12x - 2}{x(x-1)^3} = \frac{A}{x} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1}.$$

Let $x = 0$;

then $-2 = -A$; $\therefore A = 2$;

therefore by transposition we have

$$\frac{3x^3 - 10x^2 + 12x - 2}{x(x-1)^3} - \frac{2}{x} = \frac{x^3 - 4x^2 + 6x}{x(x-1)^3} = \frac{x^2 - 4x + 6}{(x-1)^3};$$

hence $$\frac{x^2 - 4x + 6}{(x-1)^3} = \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1}.$$

Let $x - 1 = z$ or $x = z + 1$, and substitute this value in both sides of the equation, and it becomes

$$\frac{z^2 - 2z + 3}{z^3} = \frac{1}{z} - \frac{2}{z^2} + \frac{3}{z^3} = \frac{B}{z^3} + \frac{C}{z^2} + \frac{D}{z};$$

$\therefore B = 3$, $C = -2$, and $D = 1$; wherefore

$$\frac{3x^3 - 10x^2 + 12x - 2}{x(x-1)^3} = \frac{2}{x} + \frac{3}{(x-1)^3} - \frac{2}{(x-1)^2} + \frac{1}{x-1}.$$

460. It is necessary that the form of the development of a function be known before it can be effected by this method, for sometimes the ordinary assumed series is not suitable, and the result of the process indicates this circumstance.

If, for instance, x enter as a multiplier into the function, as in the expression $\frac{x}{1-x} = x \times \frac{1}{1-x}$, then when $x = 0$, this function is also $= 0$, whereas the assumed series would be $= A$, although it ought also to be $= 0$; hence in such a case the series ought to be $Ax + Bx^2 + Cx^3 + \ldots$ or $x(A + Bx + Cx^2 + \ldots)$.

Again, if $\frac{1}{x}$ enter as a factor of the given function, as in $\frac{a}{x - 2x^2} = \frac{1}{x} \times \frac{a}{1 - 2x}$, then for $x = 0$, the function becomes $\frac{1}{0} \times \frac{a}{1} = \frac{a}{0} = \infty$, whereas the series would also in this case be reduced to A, although it ought to be $= \infty$; hence in such a case the series ought to be $\frac{A}{x} + B + Cx + Dx^2 +, \ldots$ or $\frac{1}{x}(A + Bx + Cx^2 + \ldots)$.

461. The simplest method, however, when the function has any factors such as the preceding, is to reject them, and find the development of the remaining part, and then to multiply this series by the factor.

462. It may be easily proved that any rational algebraic function containing one variable quantity x, which does not become either equal to zero or to infinity when $x = 0$, or any irrational function of the form $(a + bx + cx^2 + \ldots)^{\frac{r}{s}}$, in which $\frac{r}{s}$ is either positive or negative, may be developed in the form $A + Bx + Cx^2 + \ldots$

The series cannot contain either x^m or $\frac{1}{x^m}$ as a factor, for then it would become 0 or ∞, for $x = 0$, while the given functions do not become equal to 0 or ∞. The series therefore cannot contain x in all its terms, neither can x have a negative exponent in any term.

The series cannot contain a term of the form $Mx^{\frac{m}{n}}$, or any fractional power of x. For if $n = 2$, the term $Mx^{\frac{m}{2}}$ or $M\sqrt{x^m}$ would have two values corresponding to the double value $\pm\sqrt{x^m}$; while the rational function would have only one value. And if it be the above irrational function, and it be raised to the power s, then

$$(a + bx + cx^2 + \ldots)^r = (A + Bx + Cx^2 + \ldots + Mx^{\frac{m}{2}} + \ldots)^s.$$

And it is evident that the second member when involved to the power s would still contain a term of the form $Nx^{\frac{m}{2}}$, which would have two values; and hence the second member would have two values at least, while the first has only one.

When n is not $= 2$, then, whatever be its value (since it is proved in the theory of equations that the number of roots of any quantity is equal to the exponent of the root*), the term $Nx^{\frac{m}{n}}$, which would exist in the second member, after being involved to the power s, would have n values, and the second member would have at least n values, while the first member has only one. No fractional power of x therefore can exist in the development.

* This is proved in the theory of binomial equations, or equations of the form $x^n \pm a^n = 0$.

EXERCISES.

1. Develop $\frac{1}{3-x}$, ... $= \frac{1}{3} + \frac{1}{9}x + \frac{1}{27}x^2 + \ldots$

2. Convert $\frac{1}{1-2x+x^2}$ into a series, $= 1 + 2x + 3x^2 + 4x^3 + \ldots$

3. Develop $\sqrt{(1+x^2)}$, . . $= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \ldots$

4. ... $\frac{a+bx}{a'+b'x+c'x^2}$,

$$= \frac{a}{a'} - \left(\frac{b'}{a'}A - \frac{b}{a'}\right)x - \left(\frac{b'}{a'}B + \frac{c'}{a'}A\right)x^2 - \left(\frac{b'}{a'}C + \frac{c'}{a'}B\right)x^3 - \ldots$$

5. Develop $\frac{1+x}{1-2x-x^2}$,

$= 1 + (1 + 2A)x + (A + 2B)x^2 + (B + 2C)x^3 +, \ldots$ or $1 + 3x + 7x^2 + 17x^3 + 41x^4 + \ldots$

6. Resolve $\frac{x+3}{(x-1)(x+2)}$ into partial fractions,

$$= \frac{4}{3(x-1)} - \frac{1}{3(x+2)}.$$

7. ... $\frac{3x+2}{x^2-1}$ into partial fractions, $= \frac{5}{2(x-1)} + \frac{1}{2(x+1)}$.

8. ... $\frac{x+1}{x^2-7x+12}$ ... ... $= \frac{5}{x-4} - \frac{4}{x-3}$.

9. ... $\frac{1}{(x-a)(x-b)}$ ... ...

$$= \frac{1}{(a-b)(x-a)} - \frac{1}{(a-b)(x-b)}.$$

10. ... $\frac{3x+2}{x(x+)^3}$ into partial fractions,

$$= \frac{2}{x} + \frac{1}{(x+1)^3} - \frac{2}{(x+1)^2} - \frac{2}{x+1}.$$

463. If a polynomial, containing only one variable, and arranged according to its ascending powers, be equal to another similar polynomial, and if none of the coefficients be zero, the coefficients and exponents of the corresponding terms are equal,

whether the exponents be fractional or integral, positive or negative.

Let $Ax^a + Bx^b + Cx^c + \ldots = A'x^{a'} + B'x^{b'} + C'x^{c'} + \ldots$;

and if the exponents be all positive, but a not $= a'$, let $a \angle a'$; then dividing both sides by x^a,

$$A + Bx^{b-a} + Cx^{c-a} + \ldots = A'x^{a'-a} + B'x^{b'-a} + C'x^{c'-a} + \ldots$$

Now, if $x = 0$, then $A = 0$; which is contrary to the hypothesis; therefore a is not $\angle a'$; and in a similar manner it may be shewn that a' is not $\angle a$, for then would $A' = 0$; therefore $a = a'$, and if $x = 0$, $A = A'$; and hence

$$Bx^{b-a} + Cx^{c-a} + \ldots = B'x^{b'-a} + C'x^{c'-a} + \ldots;$$

and it may be similarly proved that $b - a = b' - a$, and therefore $b = b'$, and hence also $B = B'$. In the same manner it may be proved that $c = c'$ and $C = C'$; and so on.

If some of the exponents be negative, let $-a$ be the greatest one; then if both sides be multiplied by x^a, the equation would be reduced to the form

$$A + Bx^b + Cx^c + \ldots = A'x^{a'} + B'x^{b'} + C'x^{c'} + \ldots;$$

from which it may be easily shewn that the exponents and coefficients of the corresponding terms are equal.

THE BINOMIAL THEOREM.

464. The binomial theorem is a formula, by means of which any power or root of a binomial quantity may be found, without actual multiplication of the quantity successively into itself, or actual extraction of the root.

The demonstration of it in the case of integral powers may be easily effected by means of the principles of combinations.

Let three binomial quantities, $x + a$, $x + b$, $x + c$, be multiplied together, and the product will be found to be

$$\begin{array}{r|r|l} x^3 + a & x^2 + ab & x + abc. \\ + b & + ac & \\ + c & + bc & \end{array}$$

And if this product be again multiplied by the binomial $x + d$, the product is evidently

$$\begin{array}{l|l|l|l}
x^4 + a & x^3 + ab & x^2 + abc & x + abcd. \\
\quad + b & \quad + ac & \quad + abd & \\
\quad + c & \quad + bc & \quad + acd & \\
\quad + d & \quad + ad & \quad + bcd & \\
 & \quad + bd & & \\
 & \quad + cd & &
\end{array}$$

465. It is evident from the process of multiplication, that the first term consists of a power of x, whose exponent is equal to the number of binomial factors, and that its powers diminish by unity in each succeeding term. It is also evident, that the coefficient of the second term is the sum of all the second terms of the binomial; that the coefficient of the third term is the product of the second terms of the binomial, taken two and two; that the coefficient of the fourth term is the product of the same quantities, taken three and three. It is therefore very likely, on the principle of induction, that the coefficient of any term whose place is denoted by $n + 1$, or which has n terms before it, is the sum of the products of the second terms of the binomials, taken n and n together; and the last term is evidently the product of all the second terms of the binomial.

In order to prove generally the law of the coefficients, let the product of m factors be

$$= x^m + Ax^{m-1} + Bx^{m-2} + \dots. Mx^{m-n+1} + Nx^{m-n} + \dots T.$$

Here $m - n$ is the exponent of x in the $(n + 1)$th term. Let this product be multiplied by a new factor $x + k$, and the new product is

$$\begin{array}{r|r|r|r|r}
x^{m+1} + A & x^m + B & x^{m-1} + C & x^{m-2} + \dots + N & x^{m-n+1} + \dots \\
+ k & + kA & + kB & + kM & \dots + kT.
\end{array}$$

By multiplying by the new factor $x + k$, the coefficient of the second term has been increased by k; and this therefore holds true for every factor; and hence,

466. The coefficient of the second term is the sum of the second terms of the binomials.

Again, B is by hypothesis the sum of the products of the second terms of m binomials, taken two and two; and kA is the product of the second terms of the preceding m binomials by the second term k of the new binomial; and therefore $B + kA$ still continues to be the product of the second terms of all the binomial factors, taken two and two; and hence,

467. The coefficient of the third term is the sum of the pro-

ducts of the second terms of all the binomial factors, taken twc and two.

468. It may easily be shewn in the same manner, that the coefficient of the fourth term is the sum of the products of the second terms of all the binomial factors, taken three and three And, generally, since N is the sum of the products of the seconc terms of m binomials, taken n and n together, and as kM is the product of the same quantities, taken $(n-1)$ and $(n-1)$ together multiplied by k, therefore $N+kM$ is the product of the same terms of $(m+1)$ binomials, taken n and n together; that is,

469. The coefficient of any term is the sum of the products o the second terms of the binomial factors, taken n and n together where n denotes the number of the terms preceding the assumec term.

470. It is therefore manifest, that the last term is the produc of the second terms of all the binomial factors.

471. Since the law therefore of the composition of the coeffi cients, which was supposed to hold in the case of m binomia factors, continues to be true in the case of $(m+1)$ factors, i must be true generally; for it is manifestly true for two factors and therefore for three; and hence it is true for any number.

The development of $(x+a)^m$ will now be found by supposin the second terms of the binomial factors all equal. When ther are m binomial factors, of which $a, b, c, \ldots$ are the second terms if they be equal, then the coefficient of the second term is

$$A = a + b + c + \ldots = ma.$$

The second term $B = ab + ac + bc + \ldots = a^2 + a^2 +, \ldots$ a being repeated as often as there are combinations of $a, b, c, \ldots$ taken two and two; hence (452)

$$B = \frac{m(m-1)}{2}a^2.$$

The third term $C = abc + abd + bcd + \ldots = a^3 + a^3 +, \ldots$ a being repeated as often as there are combinations of $a, b, c, \ldots$ taken three and three; hence (453)

$$C = \frac{m(m-1)(m-2)}{2 \times 3}a^3;$$

and in general the coefficient of any term, as N, which is precede by n terms, is

$$N = \frac{m(m-1)(m-2)\ldots(m-n+1)}{1 \times 2 \times 3 \times \ldots n}a^n.$$

472. Hence the formula is

$$(x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{1\times 2}a^2x^{m-2}$$
$$+ \frac{m(m-1)(m-2)}{1\times 2\times 3}a^3x^{m-3} + \ldots$$
$$+ \frac{m(m-1)(m-2)\ldots(m-n+1)}{1\times 2\times 3\times\ldots n}a^nx^{m-n} + \ldots + a^m.$$

473. The term to which the coefficient N belongs is called the *general* term, because by giving to n any particular values, as $n = 1$, $n = 2$, $n = 3$, ... all the other terms may be derived from it. The term preceding the general term, for which $n - 1$ must be taken in place of n in the latter term, is evidently

$$\frac{m(m-1)(m-2)\ldots(m-n+2)}{1\times 2\times 3\times\ldots(n-1)}a^{n-1}x^{m-n+1};$$

and as $(m - n + 1)$ is the exponent of the leading quantity x in this term, and n is the number denoting the place of this term, also the coefficient of the general term is found by multiplying the preceding coefficient of the nth term by $(m - n + 1)$, and dividing it by n; therefore, generally,

474. The coefficient of any term is found by multiplying the coefficient of the preceding term by the exponent of the leading quantity in that term, and dividing by the number denoting the place of that term preceding that whose coefficient is sought.

It is also evident, that the first term is the leading quantity raised to the required power, and that the coefficient of the second term is the exponent of the required power; and that the exponents of the leading quantity diminish by unity in each succeeding term. Also, the second term of the binomial enters the second term of the power, and its exponents increase by unity in each succeeding term, and the last term is this quantity raised to the required power.

475. When the second term of the binomial is negative, the signs of the terms of the development are alternately positive and negative, or those terms are negative which contain odd powers of the negative quantity.

For in this case the second terms of all the binomial factors would be negative, and hence the odd powers of a contained in the coefficients A, C, E, ... of the even terms; namely, $(-a)$, $(-a)^3$, $(-a)^5$, ... being negative, the even terms of the series

are negative, or the terms are alternately positive and negative that is,

$$(x-a)^m = x^m - max^{m-1} + \frac{m(m-1)}{1 \times 2}a^2x^{m-2}$$
$$- \frac{m(m-1)(m-2)}{1 \times 2 \times 3}a^3x^{m-3} + \dots$$

476. The coefficients, taken in a reverse order, are the same as in a direct order—

For the last is 1; the last but one is

$$\frac{m(m-1)\dots(m-m+2)}{1 \times 2 \times 3 \times \dots (m-1)}, \text{ or } \frac{m(m-1\dots 3 \times 2}{1 \times 2 \times 3 \dots (m-1)} = m;$$

the last but two is

$$\frac{m(m-1)\dots(m-m+3)}{1 \times 2 \times 3 \dots (m-2)} = \frac{m(m-1)\dots 4 \times 3}{1 \times 2 \times 3 \dots (m-2)} = \frac{m(m-1)}{1 \times 2}$$

and the last but three would be found to be $\frac{m(m-1)(m-2)}{1 \times 2 \times 3}$ and so on.

477. The sum of the coefficients of $(x+a)^m$ is $= 2^m$.

Let $x = 1$ and $a = 1$, and the series becomes $(1+1)^m = 2^m$ $= 1 + m + \frac{m(m-1)}{2} + \dots + \frac{m(m-1)}{2} + m + 1.$

478. The sum of the coefficients of $(x-a)^m$ is $= 0$.

Let $x = 1$ and $a = 1$, then

$$(1-1)^m = 0^m = 0 = 1 - m + \frac{m(m-1)}{2} - \dots$$

EXAMPLES.

1. Find the fourth power of $b + z$.

The powers of b which enter the successive terms, beginning with the first term, are

$$b^4, b^3, b^2, b;$$

and those of z are in order, beginning with the second term,

$$z, z^2, z^3, z^4;$$

also the coefficient of the second term is 4 (474), for here $n = 4$ that of the third is $\frac{4 \times 3}{2} = 6$; of the fourth $\frac{6 \times 2}{3} = 4$; and th

fifth term is the last or z^4, as the first is b^4; hence, uniting the coefficients and the quantities that belong to the respective terms,

$$(b + z)^4 = b^4 + 4b^3z + 6b^2z^2 + 4bz^3 + z^4.$$

2. Find the fifth power of $c - y$.

The literal parts of the terms are c^5, c^4y, c^3y^2, c^2y^3, cy^4, y^5, and the coefficients are (474) 1, 5, $\frac{5 \times 4}{2} = 10$, $\frac{10 \times 3}{3} = 10$, $\frac{10 \times 2}{4} = 5$, and 1.

Hence $(c - y)^5 = c^5 - 5c^4y + 10c^3y^2 - 10c^2y^3 + 5cy^4 - y^5$.

EXERCISES.

1. Find the square of $x + a$, . . . $= x^2 + 2ax + a^2$.
2. ... cube of $y - z$, . . $= y^3 - 3y^2z + 3yz^2 - z^3$.
3. ... $(c + y)^4$, . $= c^4 + 4c^3y + 6c^2y^2 + 4cy^3 + y^4$.
4. ... $(a - x)^5$, $= a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$.
5. ... $(a + b)^6$,
$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.
6. ... $(x - y)^{10}$, $= x^{10} - 10x^9y + 45x^8y^2 - 120x^7y^3 + 210x^6y^4 - 252x^5y^5 + 210x^4y^6 - 120x^3y^7 + 45x^2y^8 - 10xy^9 + y^{10}$.

479. The binomial theorem also serves to involve any quantity consisting of two simple terms, as in the following examples.

EXAMPLES.

1. Find the cube of $2c^3 - 3z^2$.

Let $2c^3 = a$, and $3z^2 = x$, then

$$(a - x)^3 = a^3 - 3a^2x + 3ax^2 - x^3.$$

Now substitute for a and x their values, and

$$(2c^3 - 3z^2)^3 = (2c^3)^3 - 3(2c^3)^2(3z^2) + 3(2c^3)(3z^2)^2 - (3z^2)^3$$
$$= 8c^9 - 36c^6z^2 + 54c^3z^4 - 27z^6.$$

2. Find the fourth power of $2y - \frac{x^2}{z^3}$.

Let $2y = a$, and $\frac{x^2}{z^3} = b$, then

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4, \text{ or}$$

$$\left(2y - \frac{x^2}{z^3}\right)^4 = (2y)^4 - 4(2y)^3\frac{x^2}{z^3} + 6(2y)^2\left(\frac{x^2}{z^3}\right)^2 - 4(2y)\left(\frac{x^2}{z^3}\right)^3$$

$$+ \left(\frac{x^2}{z^3}\right)^4 = 16y^4 - 32\frac{x^2y^3}{z^3} + 24\frac{x^4y^2}{z^6} - 8\frac{x^6y}{z^9} + \frac{x^8}{z^{12}}.$$

EXERCISES.

1. $(3a - 4x^2)^2$, ... $= 9a^2 - 24ax^2 + 16x^4$.

2. $\left(\frac{2x^3}{y^2} + 3z^2\right)^3$, . . $= \frac{8x^9}{y^6} + \frac{36x^6z^2}{y^4} + \frac{54x^3z^4}{y^2} + 27z^6$.

3. $\left(3y^2 - \frac{z^3}{2}\right)^4$, . $= 81y^8 - 54y^6z^3 + \frac{27y^4z^6}{2} - \frac{3y^2z^9}{2} + \frac{z^{12}}{16}$.

4. $(2x - 3y)^5$,

$= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$.

5. $\left(x - \frac{1}{x}\right)^7$, $= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}$.

480. The preceding development of $(x+a)^m$ is true, whether m be integral or fractional, positive or negative; that is, for $(x+a)^m$, $(x+a)^{-m}$, $(x+a)^{\frac{m}{n}}$, or $(x+a)^{-\frac{m}{n}}$.

In order to demonstrate this, let $(x+a)^m$ be put under the form $x^m\left(1+\frac{a}{x}\right)^m$, and let $\frac{a}{x} = y$; then if the development of $(1+y)^m$ be found, and multiplied by x^m, the result will be the development of $(x+a)^m$.

It appears (462) that the equation

$$(1+y)^{\frac{m}{n}} = 1 + Ay + By^2 + Cy^3 + Dy^4 + \ldots \qquad [1],$$

may be assumed, its first term being 1, because when $y = 0$, the

first member becomes 1, and also the second. Hence also if z be any quantity,

$$(1+z)^{\frac{m}{n}} = 1 + Az + Bz^2 + Cz^3 + Dz^4 +, \ldots$$

in which the coefficients must evidently be the same as in the former series, for both equations are true for any values of y and z, and therefore when $y = z$.

Now, let $(1+y)^{\frac{1}{n}} = u$, and $(1+z)^{\frac{1}{n}} = v$,

then $(1+y)^{\frac{m}{n}} = u^m$, and $(1+z)^{\frac{m}{n}} = v^m$; also

$$\begin{aligned} u^m - v^m = {} & A(y-z) + B(y^2-z^2) + C(y^3-z^3) \\ & + D(y^4 - z^4) + \quad \ldots \quad \ldots \quad [2]. \end{aligned}$$

But $1 + y = u^n$, $1 + z = v^n$; then $u^n - v^n = y - z$; hence

$$\begin{aligned} \frac{u^m - v^m}{u^n - v^n} = {} & \frac{1}{y-z}\{A(y-z) + B(y^2-z^2) + C(y^3-z^3) \\ & + D(y^4-z^4) + \ldots\} \quad \ldots \quad \ldots \quad [3], \end{aligned}$$

and (87)

$$\begin{aligned} u^m - v^m &= (u-v)(u^{m-1} + u^{m-2}v + u^{m-3}v^2 + \ldots + uv^{m-2} + v^{m-1}) \\ &= (u-v)M, \end{aligned}$$

and

$$\begin{aligned} u^n - v^n &= (u-v)(u^{n-1} + u^{n-2}v + u^{n-3}v^2 + \ldots + uv^{n-2} + v^{n-1}) \\ &= (u-v)N; \end{aligned}$$

assuming M and N respectively for the polynomial factors.

Hence $$\begin{aligned} \frac{M}{N} = \frac{u^m - v^m}{u^n - v^n} = {} & A + B(y+z) + C(y^2 + yz + z^2) \\ & + D(y^3 + y^2z + yz^2 + z^3) + \quad \ldots \quad \ldots \quad [4]. \end{aligned}$$

And as this equation holds true for any values of y and z, it does so for $y = z$, for which $1 + y = 1 + z$, or $u = v$; and therefore $M = mu^{m-1}$, $N = nu^{n-1}$, and by [4],

$$\frac{mu^{m-1}}{nu^{n-1}} = A + 2By + 3Cy^2 + 4Dy^3 + \quad \ldots \quad \ldots \quad [5],$$

or $$\frac{mu^m}{n} = u^n(A + 2By + 3Cy^2 + 4Dy^3 + \ldots).$$

Substituting for u^m and u^n their values $(1+y)^{\frac{m}{n}}$ and $1+y$,

$$\frac{m}{n}(1+y)^{\frac{m}{n}} = (1+y)(A + 2By + 3Cy^2 + 4Dy^3 + \dots);$$

and therefore by equation [1],

$$\frac{m}{n} + \frac{m}{n}Ay + \frac{m}{n}By^2 + \frac{m}{n}Cy^3 = \begin{array}{l|l|l|l} A + 2B & y + 3C & y^2 + 4D & y^3 + \dots \\ \quad + A & \; + 2B & \; + 3C & \; + \dots \end{array}$$

And equating the corresponding coefficients,

$$\frac{m}{n} = A,\ \frac{m}{n}A = 2B + A,\ \frac{m}{n}B = 3C + 2B,\ \frac{m}{n}C = 4D + 3C, \dots$$

whence

$$A = \frac{m}{n},\ B = \frac{A}{2}\left(\frac{m}{n} - 1\right),\ C = \frac{B}{3}\left(\frac{m}{n} - 2\right),\ D = \frac{C}{4}\left(\frac{m}{n} - 3\right), \dots$$

and hence the law of the composition of the coefficients is evident. Let N be the coefficient of the $(r+1)$th term, and M that of the rth, then it is evident that

$$\frac{m}{n}M = rN + (r-1)M, \text{ whence } N = \frac{M}{r}\left(\frac{m}{n} - r + 1\right).$$

481. The coefficients being now found, the development is known. By making $n = 1$ in the preceding investigation, the theorem will be established for an integral exponent—namely, m; and it may be remarked, that the preceding demonstration is independent of induction.

For the case of a fractional negative exponent, as $-\frac{m}{n}$, let m be changed into $-m$, in the demonstration, as far as equation [2] inclusive, then the first member of this equation becomes

$$u^{-m} - v^{-m} = \frac{1}{u^m} - \frac{1}{v^m} = \frac{v^m - u^m}{u^m v^m}; \text{ and}$$

$$\frac{u^{-m} - v^{-m}}{u^n - v^n} = -\frac{1}{u^m v^m} \cdot \frac{u^m - v^m}{u^n - v^n}.$$

Hence the first member of [3] becomes $-\frac{1}{u^m v^m} \cdot \frac{u^m - v^m}{u^n - v^n}$; and that of [4], $-\frac{1}{u^m v^m} \cdot \frac{M}{N}$; and when $y = z$, and therefore $u = v$

$-\dfrac{1}{u^m v^m} = -\dfrac{1}{u^{2m}}$; and hence the first side of [5] becomes

$-\dfrac{1}{u^{2m}} \cdot \dfrac{mu^{m-1}}{nu^{n-1}} = \dfrac{-mu^{-m-1}}{nu^{n-1}}$; and hence the equation itself is now

$$\frac{-mu^{-m-1}}{nu^{n-1}} = A + 2By + 3Cy^2 + 4Dy^3 + ; \ldots$$

an equation which differs from [5] only in the sign of m, and therefore the values of the coefficients will be found from the preceding values, by merely changing the sign of m.

Hence, whatever be the value or the sign of m, the formula

$$(1+y)^m = 1 + my + \frac{m(m-1)}{2}y^2 + \frac{m(m-1)(m-2)}{2 \times 3}y^3 + \ldots$$

holds true; therefore substituting $\dfrac{a}{x}$ for y,

$$x^m\left(1+\frac{a}{x}\right)^m, \text{ or } (x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{2}a^2x^{m-2}$$
$$+ \frac{m(m-1)(m-2)}{2 \times 3}a^3x^{m-3} + \ldots$$

Or taking X and Y for x and a,

$$(X+Y)^m = X^m + mX^{m-1}Y + \frac{m(m-1)}{2}X^{m-2}Y^2$$
$$+ \frac{m(m-1)(m-2)}{2 \times 3}X^{m-3}Y^3 + \ldots$$

482. If the coefficients of the successive terms, beginning with the second, be respectively A, B, C, D, ... then

$$(Y+X)^m = Y^m\left\{1 + A\frac{X}{Y} + B\left(\frac{X}{Y}\right)^2 + C\left(\frac{X}{Y}\right)^3 + \ldots\right\} \ldots \quad \text{[I.]}$$

$$(Y+X)^{-m} = \frac{1}{Y^m}\left\{1 + A\frac{X}{Y} + B\left(\frac{X}{Y}\right)^2 + C\left(\frac{X}{Y}\right)^3 + \ldots\right\} \ldots \quad \text{[II.]}$$

$$(Y+X)^{\frac{m}{n}} = Y^{\frac{m}{n}}\left\{1 + A\frac{X}{Y} + B\left(\frac{X}{Y}\right)^2 + C\left(\frac{X}{Y}\right)^3 + \ldots\right\} \ldots \quad \text{[III.]}$$

$$(Y+X)^{-\frac{m}{n}} = \frac{1}{Y^{\frac{m}{n}}}\left\{1 + A\frac{X}{Y} + B\left(\frac{X}{Y}\right)^2 + C\left(\frac{X}{Y}\right)^3 + \ldots\right\} \ldots \quad \text{[IV.]}$$

And the coefficients for these four cases are in order,

$$A = m,\ B = A\frac{m-1}{2},\ C = B\frac{m-2}{3},\ D = C\frac{m-3}{4},\ \ldots \quad [\text{I.}]$$

$$A = -m,\ B = -A\frac{m+1}{2},\ C = -B\frac{m+2}{3},\ D = -C\frac{m+3}{4},\ \ldots \quad [\text{II.}]$$

$$A = \frac{m}{n},\ B = A\frac{m-n}{2n},\ C = B\frac{m-2n}{3n},\ D = C\frac{m-3n}{4n}, \ldots [\text{III.}]$$

$$A = -\frac{m}{n},\ B = -A\frac{m+n}{2n},\ C = -B\frac{m+2n}{3n},\ D = -C\frac{m+3n}{4n}, \ldots [\text{IV.}]$$

483. As the series of coefficients in the second and fourth lines are already adapted to the negative exponents, the values of m and n in them must be taken with the positive sign.

EXAMPLES.

1. Find the development of $\frac{a}{(c-y)^3}$, or $a(c-y)^{-3}$.

By [II.] $m = 3$, $Y = c$, $X = -y$, $\frac{X}{Y} = -\frac{y}{c}$, $\frac{1}{Y^m} = \frac{1}{c^3}$; and

$$A = -3,\ B = 3 \times \frac{3+1}{2} = 6,\ C = -6 \times \frac{5}{3} = -10,$$

$$D = 10 \times \frac{6}{4} = 15 \ldots$$

Hence

$$\frac{a}{(c-y)^3} = \frac{a}{c^3}\{1 + \frac{3y}{c} + \frac{6y^2}{c^2} + \frac{10y^3}{c^3} + \frac{15y^4}{c^4} + \ldots\},$$

or
$$\frac{a}{(c-y)^3} = \frac{a}{c^3}\{1 + \frac{3y}{c} + \frac{3 \times 4}{2} \cdot \frac{y^2}{c^2} + \frac{3 \times 4 \times 5}{2 \times 3} \cdot \frac{y^3}{c^3}$$

$$+ \frac{3 \times 4 \times 5 \times 6}{2 \times 3 \times 4} \cdot \frac{y^4}{c^4} + \ldots\}$$

2. Develop $\sqrt[5]{(a^2 - x^2)}$ or $(a^2 - x^2)^{\frac{1}{5}}$.

By [III.] $m = 1$, $n = 5$, $\frac{m}{n} = \frac{1}{5}$, $Y = a^2$, $X = -x^2$,

$$\frac{X}{Y} = -\frac{x^2}{a^2},\ Y^{\frac{m}{n}} = (a^2)^{\frac{1}{5}} = a^{\frac{2}{5}};$$

also $A = \frac{1}{5}$, $B = \frac{1}{5} \times \frac{-4}{10} = -\frac{2}{25}$, $C = -\frac{2}{25} \times \frac{-9}{15} = \frac{6}{125}$ ...

Hence $\sqrt[5]{(a^2 - x^2)} = a^{\frac{2}{5}}(1 - \frac{x^2}{5a^2} - \frac{2x^4}{25a^4} - \frac{6x^6}{125a^6} - \ldots)$.

3. Find the development of $\sqrt[3]{\frac{a^2}{(a^2 + x^2)^2}}$ or $a^{\frac{2}{3}}(a^2 + x^2)^{-\frac{2}{3}}$.

Here by [IV.], $m = 2$, $n = 3$, $X = x^2$, $Y = a^2$, $\frac{X}{Y} = \frac{x^2}{a^2}$, $\frac{1}{Y^{\frac{m}{n}}} = \frac{1}{a^{\frac{4}{3}}}$,

and $A = -\frac{2}{3}$, $B = \frac{2}{3} \cdot \frac{5}{6}$, $C = -\frac{2}{3} \cdot \frac{5}{6} \cdot \frac{8}{9}$, $D = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{8}{9} \cdot \frac{11}{12}$, &c. ...

Hence, simplifying the coefficients,

$$a^{\frac{2}{3}}(a^2 + x^2)^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}(1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6} + \frac{110x^8}{243a^8} - \ldots).$$

4. Develop, in the form of a series, $\sqrt[5]{28}$.

Assume $28 = 32 - 4$, then $\sqrt[5]{28} = \sqrt[5]{(32 - 4)}$; and hence by [III.], $\frac{m}{n} = \frac{1}{5}$, $Y = 32$, $X = -4$, $\frac{X}{Y} = -\frac{1}{8}$, and $Y^{\frac{m}{n}} = \sqrt[5]{32} = 2$; in order that $\sqrt[5]{Y}$ might be an integer, 32 was assumed as the first term of the binomial, and consequently -4 is the second term. The coefficients are the same as in example 2; hence

$$\sqrt[5]{28} = \sqrt[5]{(32 - 4)} = 2(1 - \frac{1}{5} \cdot \frac{1}{8} - \frac{1}{5} \cdot \frac{4}{10} \cdot \frac{1}{8^2} - \frac{1}{5} \cdot \frac{4}{10} \cdot \frac{9}{15} \cdot \frac{1}{8^3}$$
$$- \frac{1}{5} \cdot \frac{4}{10} \cdot \frac{9}{15} \cdot \frac{14}{20} \cdot \frac{1}{8^4} - \ldots).$$

5. Develop $\sqrt{\frac{a - x}{a + x}}$.

As $\sqrt{\frac{a - x}{a + x}} = \left(\frac{a - x}{a + x}\right)^{\frac{1}{2}}$, or $(a - x)^{\frac{1}{2}}(a + x)^{-\frac{1}{2}}$.

Multiply both terms by $\sqrt{a + x}$, and it becomes

$$\sqrt{\frac{a^2 - x^2}{(a + x)^2}} = \frac{1}{a + x}\sqrt{a^2 - x^2} = \frac{a}{a + x}\sqrt{1 - \frac{x^2}{a^2}}; \text{ hence}$$

$$\sqrt{\frac{a - x}{a + x}} = \frac{a}{a + x}\{1 - \frac{1}{2} \cdot \frac{x^2}{a^2} - \frac{1}{2^2} \cdot \frac{1}{2} \cdot \frac{x^4}{a^4} - \frac{1}{2^3} \cdot \frac{1 \times 3}{2 \times 3} \cdot \frac{x^6}{a^6} - \text{\&c.}\}$$

EXERCISES.

1. Develop $\frac{1}{(1-x)^2}$, . . $= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$

2. ... $\frac{a^2}{(a+x)^2}$, . $= 1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \frac{5x^4}{a^4} - \ldots$

3. Find the value of $\frac{1}{(a-x)^3}$ in a series,

$$= \frac{1}{a^3}\left(1 + \frac{3}{1}\cdot\frac{x}{a} + \frac{3}{1}\cdot\frac{4}{2}\cdot\frac{x^2}{a^2} + \frac{3}{1}\cdot\frac{4}{2}\cdot\frac{5}{3}\cdot\frac{x^3}{a^3} + \frac{3}{1}\cdot\frac{4}{2}\cdot\frac{5}{3}\cdot\frac{6}{4}\cdot\frac{x^4}{a^4} + \ldots\right).$$

4. Develop $\sqrt[3]{(a^3 - x)}$, $= a - \frac{x}{3a^2} - \frac{x^2}{9a^5} - \frac{5x^3}{81a^8} - \frac{10x^4}{243a^{11}} - \ldots$

5. ... $\sqrt[3]{(1-x^3)}$, . . $= 1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81} - \ldots$

6. ... $\sqrt[3]{6}$ or $\sqrt[3]{(8-2)}$, $= 2\left(1 - \frac{1}{3}\cdot\frac{1}{4} - \frac{1}{9}\cdot\frac{1}{4^2} - \frac{5}{81}\cdot\frac{1}{4^3} - \ldots\right)$.

484. Trinomials and polynomials of few terms may be easily developed by means of this theorem.

1. To develop any powers of $a + b + c$, assume $a + b$, or $b + c$, equal to a single letter; thus, if $b + c = d$, then

$$(a + b + c)^2 = (a + d)^2 = a^2 + 2ad + d^2 = a^2 + 2a(b + c)$$
$$+ (b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2;$$

hence $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.

Or if it be assumed that $a + b = p$, then

$$(a + b + c)^2 = (p + c)^2 = p^2 + 2pc + c^2 = (a + b)^2$$
$$+ 2(a + b)c + c^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2.$$

2. Similarly, if $a + b + c = p$, then

$$(a + b + c + d)^3 = (p + d)^3 = p^3 + 3p^2d + 3pd^2 + d^3$$
$$= (a + b + c)^3 + 3(a + b + c)^2d + 3(a + b + c)d^2 + d^3.$$

3. So $(a + b + c + d)^4 = (a + b + c)^4 + 4(a + b + c)^3d +$
$6(a + b + c)^2d^2 + 4(a + b + c)d^3 + d^4$.

Or if $a + b = p$ and $c + d = q$,

$$(a + b + c + d)^4 = (p + q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4,$$

and by substituting for p and q their values, the development of the given quantity is found in a new form; but by resolving the compound terms into their component simple terms, as $4(a + b + c)d^3 = 4ad^3 + 4bd^3 + 4cd^3$, the same result will be found for the value of $(a + b + c + d)^4$ by different methods.

EXPONENTIAL THEOREM.

485. An *exponential function* is a quantity with a variable exponent.

Thus, in the equation $a^x = y$, x is a variable exponent, and a^x is an exponential function of x.

486. To develop a^x, assume $a = 1 + b$, then (481) $a^x = (1 + b)^x =$

$$1 + xb + \frac{x(x-1)}{2}b^2 + \frac{x(x-1)(x-2)}{2 \times 3}b^3 + \ldots \qquad [1].$$

Hence it is evident that a^x may be developed in a series containing only integral and positive powers of x, and also that the first term is 1, and the coefficients are functions of b or $(a - 1)$; hence let

$$a^x = 1 + kx + Ax^2 + Bx^3 + Cx^4 + \ldots \qquad [2].$$

By inspection of the series [1], it is evident that the coefficients of x in the different terms are

$$k = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \frac{b^5}{5} - \ldots \qquad [3],$$

in which b or $a - 1$ is known, and hence k is known.

If in [2] x be assumed $= z$, then

$$a^z = 1 + kz + Az^2 + Bz^3 + \ldots;$$

hence, subtracting the equation [2] from this,

$$a^z - a^x = k(z - x) + A(z^2 - x^2) + B(z^3 - x^3) + \ldots$$

If now z be assumed $= x + h$, the first member of the last

equation becomes $a^{x+h} - a^x = a^x(a^h - 1)$; and if in [2] h be substituted for x, then $a^h - 1 = kh + Ah^2 + \ldots$;

and $$a^x(a^h - 1) = a^x(kh + Ah^2 + Bh^3 + \ldots).$$

Now, the first members of the two last equations are equal, for $a^z = a^{x+h}$; and hence the second members are also equal, and are divisible by $z - x = h$; hence, by dividing,

$$a^x(k + Ah + Bh^2 + \ldots) = k + A(z + x) + B(z^2 + zx + x^2) + \ldots;$$

and as h is arbitrary, let it $= 0$, and then $z = x$, and the first member is reduced to ka^x; and taking the value in [2] instead of a^x, then $ka^x =$

$$k(1 + kx + Ax^2 + Bx^3 + \ldots) = k + 2Ax + 3Bx^2 + 4Cx^3 + \ldots$$

Therefore (459) $2A = k^2$, $3B = Ak$, $4C = Bk$, $5D = Ck \ldots$;

and hence $A = \frac{k^2}{2}$, $B = \frac{k^3}{2 \times 3}$, $C = \frac{k^4}{2 \times 3 \times 4} \ldots$

And substituting these values for the coefficients in [2], it becomes

$$a^x = 1 + kx + \frac{k^2x^2}{2} + \frac{k^3x^3}{2 \times 3} + \ldots \frac{k^nx^n}{2 \times 3 \times 4 \ldots \times n} \ldots \quad [4].$$

Any term of this series is found by multiplying the preceding term by $\frac{kx}{n}$, where n denotes the number of terms that precede the term sought; and as kx is constant for any particular values of k and x, while n is continually increasing, the quantity $\frac{kx}{n}$ will become indefinitely small, and the series will therefore *converge;* that is, its terms will diminish indefinitely.

487. The series [3] gives the value of k in terms of b or of $a - 1$, for (486) $a = 1 + b$; and in order to find that value of a which would make $k = 1$, substitute 1 for k in [4]; and as this equation exists for any value of x, let also $x = 1$, as its value is arbitrary; then

$$a = 1 + 1 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{2 \times 3 \times 4 \times 5} + \ldots$$

Now if 13 terms of this series be taken, and the decimals be carried to 10 places, the value of a, which corresponds to $k = 1$, will be found.

Let e represent this value of a, then $e = 2{\cdot}7182818284$.

The values of the terms of the preceding series are easily found; for if $l =$ the fifth term, l being expressed as a decimal fraction, then the sixth term is $= \frac{l}{5}$; if l denote the sixth, then $\frac{l}{6} =$ the seventh; and so on.

LOGARITHMS.

488. The *logarithm* of a number is the exponent of the power to which another given number must be raised, in order to produce the former number.

Thus, if a be a given number, and y any other, then if $y = a^x$, the exponent x is the logarithm of y. The exponent may be fractional, as $a^{\frac{x}{z}} = y$, and then $a^x = y^z$. And whatever be the numbers a and y, provided $a > 1$, it is evident that some exponent $\frac{x}{z}$ will exist, such that $a^{\frac{x}{z}} = y$, or $a^x = y^z$, at least approximately, and even to any degree of approximation that may be required. The fact, that series may be found, as is afterwards done, to express the value of the logarithm of any number in terms of that number, proves the possibility of the preceding equation.

489. A *system* of logarithms consists of the logarithms of the natural series of numbers 1, 2, 3, 4, ... carried to any arbitrary term.

490. The given number, the exponents of whose powers are the logarithms, is called the *base* of the system.

Thus, in the common system of logarithms, 10 is the base. The logarithms of the powers of 10—namely, 100, 1000, 10,000, ... are evidently 2, 3, 4, ... for $10^2 = 100$, $10^3 = 1000$ ...; hence if l denote logarithm, then $2 = l100$, $3 = l1000$, ... Were 2 the base of a system; then since $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, ... in this system $2 = l4$, $3 = l8$, $4 = l16$.

GENERAL PROPERTIES OF LOGARITHMS.

THEOREMS.

I. The logarithm of the product of two numbers is equal to the sum of the logarithms of these numbers.

For let $a^x = y$, and $a^{x'} = y'$, then for the base a, $x = ly$, and $x' = ly'$. And $a^x \times a^{x'} = yy'$, or $a^{x+x'} = yy'$; and hence $x + x' = lyy'$, or $lyy' = ly + ly'$.

II. The logarithm of the quotient of two numbers is equal to the difference of their logarithms.

For $a^x \div a^{x'} = \frac{y}{y'}$, or $a^{x-x'} = \frac{y}{y'}$; and hence $x - x' = l\frac{y}{y'}$, or $l\frac{y}{y'} = ly - ly'$.

III. The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

If $a^x = y$, then $a^{nx} = y^n$; and therefore $nx = ly^n$, or $ly^n = nly$.

IV. The logarithm of a root of a number is equal to the logarithm of the number divided by the exponent of the root.

Let $ax = y$, then $a^{\frac{x}{n}} = y^{\frac{1}{n}}$; hence $\frac{x}{n} = ly^{\frac{1}{n}}$, or $l\sqrt[n]{y} = \frac{ly}{n}$.

V. The logarithm of the base of any system of logarithms taken in that system is unity, and that of unity in any system is zero.

a^1 and a^0 cannot be considered as powers of a; but (Theo. II.) $ly - ly' = x - x'$, and when $x = m + 1$ and $x' = m$, then

$$ly - ly' = la = m + 1 - m = 1;$$

also when $$x = m \text{ and } x' = m$$

$$ly - ly' = la^0 = l1 = m - m = 0.$$

VI. The logarithm of a number is positive or negative, according as the number is greater or less than unity.

Let the number be $\frac{y}{y'}$, and let $ax = y$, and $a^{x'} = y'$, then

$$a^{x-x'} = \frac{y}{y'}, \text{ or } x - x' = l\frac{y}{y'}.$$

But if $\frac{y}{y'} > 1$, then is $y > y'$; therefore $x > x'$, and $l\frac{y}{y'}$ is positive. But when $\frac{y}{y'} < 1$, then is $y' > y$, and therefore $x' > x$ and $l\frac{y}{y'}$ is negative.

When $y' = 1$, the number is a whole number, and $x' = 0$; and hence $x = ly$ is positive, as in the first case, which includes this one. Also, when $y = 1$, $x = 0$, and $x - x' = -x'$, or $l\frac{1}{y'}$ is negative, as in the second case, which includes this one.

VII. The logarithms of two numbers, taken in any system, are proportional to their logarithms in any other system.

Let a and b be the bases of two systems, y and y' two numbers, x and x' their logarithms for the former base, z and z' their logarithims for the latter base, then

$$a^x = y,\ a^{x'} = y', \text{ and } b^z = y,\ b^{z'} = y'.$$

Hence $a^x = b^z$, and $a^{x'} = b^{z'}$, and denoting by L logarithms in any system,

$$xLa = zLb, \text{ and } x'La = z'Lb;$$

hence $$x : z = Lb : La, \text{ but } x' : z' = Lb : La,$$

$$\therefore x : z = x' : z'.$$

VIII. The logarithms of the same number in two different systems are inversely as the logarithms of their bases in these system.

$$\text{Let } a^x = y, \text{ and } b^{x'} = y;$$

therefore $$a^x = b^{x'}, \text{ and hence } xLa = x'Lb,$$

$$\therefore x : x' = Lb : La.$$

In the series [4] of article (486), x being arbitrary, it may be assumed in any particular system $= \frac{1}{k}$, or $kx = 1$, and then the following relation, which exists between a, k, and e, is easily deduced. The first member then becomes $a^{\frac{1}{k}}$, and the second is e (487); hence

$$a^{\frac{1}{k}} = e, \text{ or } e^k = a.$$

491. There is a system of logarithms for which e is assumed as the base; this system is called the *natural system.**

Let logarithms in the natural system be denoted by l, those in the system whose base is a by L, then let

$$a^x = y, \text{ and then } xla = ly, \text{ or } x = \frac{ly}{la};$$

$$\text{but } x = Ly; \text{ hence } Ly = ly \cdot \frac{1}{la}.$$

The quantity la by which the natural logarithm of a number must be divided to produce its logarithm in the system whose base is a, is equal to the quantity k for this system; for

$$e^k = a, \text{ and hence } kle = la, \text{ or (Theo. V.) } k = la.$$

* It is also frequently called the *Napierian system*, from the inventor of logarithms, Lord Napier, who, in 1614. published Tables constructed on this system; and which is sometimes, though improperly, termed the *hyperbolic* system, from a property of the hyperbola, which, however, refers equally to every system.

Hence the quantity k for any system is equal to the natural logarithm of its base. Also,

$$kLe = La = 1 \text{ (Theo. V.), or } k = \frac{1}{Le},$$

and if L' denote logarithms in any system whatever, it may be similarly shewn that

$$kL'e = L'a, \text{ and } k = \frac{L'a}{L'e}.$$

The quantity $\frac{1}{k}$, or its equals $\frac{1}{la}$ or Le, which is constant for the same system, is called the *modulus* of this system, in reference to the natural system; and it may therefore properly be called the *natural modulus* of the system. Hence

492. The natural modulus of any system of logarithms is the reciprocal of the natural logarithm of its base; or the reciprocal of the quantity k; or it is equal to the logarithm of the base of the natural system taken in the former system.

493. The logarithm of a number in any system is equal to the product of the natural logarithm of this number, and the natural modulus of the system.

494. Let M denote the natural modulus of any system, whose base is a, and y any number, then

$$M = \frac{1}{la} = \frac{1}{k} = Le,$$

and (491) $$Ly = Mly.$$

Hence if the system of natural logarithms were first computed, those of any other system may be found, by multiplying the former logarithms by the natural modulus of the latter system.

495. The natural modulus of a system whose base is a; that is, the quantity $\frac{1}{k}$, has therefore three different expressions for its value. Let l, L, and L', represent logarithms, as in art. (491), then

$$M = \frac{1}{k} = \frac{1}{la} = Le = \frac{L'e}{L'a}.$$

SERIES FOR THE COMPUTATION OF LOGARITHMS.

496. The equation $e^k = a$ affords this result $L'a = kL'e$, in which $a = 1 + b$, and k is the series [3] art. (486); hence taking x instead of b, as b may have any value,

$$L'(1 + x) = L'e(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots) \quad \ldots \quad [1].$$

If instead of logarithms denoted by L', those denoted by L be

taken (491), then $L'e$ becomes $Le = M =$ the natural modulus of the system whose base is a, the above equation becomes

$$L(1+x) = M(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots).$$

And changing x into $-x$, it becomes

$$L(1-x) = -M(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} +).$$

Subtracting the latter from the former, gives

$$L\left(\frac{1+x}{1-x}\right) = 2M(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots).$$

Assume $\frac{1+x}{1-x} = 1 + \frac{z}{n}$, whence $x = \frac{z}{2n+z}$, and $\frac{1+x}{1-x} = \frac{n+z}{n}$;

and hence $L\left(\frac{1+x}{1-x}\right) = L\left(\frac{n+z}{n}\right) = L(n+z) - Ln$; therefore the preceding series becomes

$$L(n+z) = Ln +$$

$$2M\{\frac{z}{2n+z} + \frac{1}{3}\left(\frac{z}{2n+z}\right)^3 + \frac{1}{5}\left(\frac{z}{2n+z}\right)^5 + \ldots\} \quad \ldots \quad [2].$$

When the logarithm of the member n therefore is known, that of $n+z$ will be found by this series, which will always be convergent, and will be the more so the greater n is compared with z.

Let $z = 1$, then the logarithm of $n+1$ will be

$$L(n+1) = Ln +$$

$$2M\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \ldots\} \quad \ldots \quad [3],$$

which is convergent, even in the case of $n = 1$.

497. This series will be of the simplest form when the modulus $M = 1$. But $M = Le$, and when $Le = 1$, e must be the base of the system (THEO. V.) It was stated in article (491) that the system whose base is e, is called the natural system, which is an appropriate term for a system that has the simplest modulus. It was shewn in article (487) that $e = 2{\cdot}7182818$, and that this is that value of the base which makes $k = 1$; and hence also $\frac{1}{k} = 1 = M$ (492). The formula for natural logarithms, therefore, is

$$l(n+1) = ln +$$

$$2\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \ldots\} \quad \ldots \quad [4].$$

498. For the common system of logarithms, or those of Briggs, $a = 10$, and $M = \frac{1}{l10}$. The natural logarithm of 10 will be found by first computing that of 2 and 5 by the preceding series. If $n = 1$, $ln = l1 = 0$, and

$$l(1+1) = l2 = 2\left(\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \frac{1}{7}\cdot\frac{1}{3^7} + \ldots\right).$$

The 8th term of this series is less than 1 in the 8th decimal place, or less than ·00000001, and the preceding 7 terms carried to 8 decimal places, give

$$l2 = \cdot 69314718,$$

which is correct to the 8th decimal place. Again, $l5$ will be found by making $n = 4$, for then $ln = l4 = l2^2 = 2l2$, which is already known, and

$$l(4+1) = l5 = 2l2 + 2\left(\frac{1}{9} + \frac{1}{3}\cdot\frac{1}{9^3} + \frac{1}{5}\cdot\frac{1}{9^5} + \ldots\right).$$

The first three terms of this series are sufficient to give a result correct to the 7th decimal place, and the first four terms give

$$\begin{array}{lr} & \cdot 22314355 \\ \text{and} \quad l4 = 2l2 = & 1\cdot 38629436 \\ \hline \text{hence} \quad l5 = & 1\cdot 60943791 \\ \text{Also} \quad l2 = & \cdot 69314718 \\ \hline \end{array}$$

therefore $l5 + l2 = l10 = 2\cdot 30258509 =$ the reciprocal of the modulus of the common system.

Hence $$M = \frac{1}{l10} = \frac{1}{2\cdot 30258509} = \cdot 43429448.$$

499. If, then, natural logarithms be multiplied by this number, the products will be the common logarithms of the same numbers. Denoting common logarithms by L, then $L2$ and $L5$ may be found from $l2$ and $l5$; thus,

$$L2 = Ml2 = \cdot 3010300$$
$$L5 = Ml5 = \cdot 6989700$$

500. Were it required to find the natural logarithm of a number whose common logarithm is given, it would only be necessary to divide the former by $M = \frac{1}{l10}$, or to multiply by $\frac{1}{M} = l10 = 2\cdot 30258509$; but $l10 = \frac{1}{Le}$, or $Le = \frac{1}{l10}$; hence $L2\cdot 7182818 = \cdot 43429448$.

COMMON LOGARITHMS.

501. The logarithm of any power of 10, the base of the system of common logarithms, is the exponent of the power.

Let a number $m = 10^n$, then $Lm = nL10 = n \times 1 = n$.

Thus, let $m = 10^2 = 100$, then $L100 = 2$, so $L1000 = 3$, $L10000 = 4$, &c.

502. Hence the logarithms of numbers between 100 and 1000 are between 2 and 3, or $= 2 +$ a fraction; those between 1000 and 10000 are between 3 and 4, or $= 3 +$ a fraction, and so on; therefore, logarithms of all numbers that are not exact powers of 10, consist of an integer and a fractional part.

503. The integral part of a logarithm is called its *index*.

The index of the logarithm of a number is a unit less than the number of integral figures in the number.

504. The decimal part of the logarithm of a number is the same as that of the product of this number by any power of 10.

For if $m = n \times 10^r$, $Lm = Ln + r$, for $L10^r = rL10 = r$ or the whole number r being added to Ln, gives Lm; therefore the fractional part of Lm is the same as that of Ln.

So $12432 = 124{\cdot}32 \times 10^2$; hence $L12432 = L124{\cdot}32 + 2$; hence if 2 be added to $L124{\cdot}32$, it gives $L12432$; therefore the fractional parts are the same, and only the indices differ, that of the former being 2, and of the latter $2 + 2 = 4$.

505. The logarithm of any number less than 1 is negative.

506. Thus, let ·0345 be the number, then

$$\cdot 0345 \times 100 = 3{\cdot}45,$$

and $\quad L{\cdot}0345 + L10^2 = L3{\cdot}45$, or $L{\cdot}0345 + 2 = L3{\cdot}45$,

or $\quad L{\cdot}0345 = L3{\cdot}45 - 2 = 0{\cdot}537819 - 2 = \bar{2}{\cdot}537819$;

for $L3{\cdot}45$ has 0 for its index (503). The logarithm of 3·45 is taken from the tables. The logarithm of ·0345 therefore has the negative index 2, the sign being, as usual, placed *over* it, because the fractional part ·537819 is positive. If the latter part be taken from 2, the remainder is 1·462181, which is all negative; but the former mode of expressing the logarithm is commonly used, and the fact of the exponent only being negative, must be observed in calculating with logarithms. Were the number ·00345, then since

it multiplied by 1000, gives 3·45, the index would be $\bar{3}$, and the logarithm $= \bar{3}{\cdot}537819$; and, in general,

507. The index of the logarithm of a fractional number is negative, and is one greater than the number of prefixed ciphers.

The *arithmetical complement* of a number is found by taking its first figure from 10, and all the rest from 9, and then prefixing $\bar{1}$.

Thus, the arithmetical complement of 573 is $= \bar{1}427$, which is $= -1000 + 427 = -573$; and hence to add $\bar{1}427$ is the same as to subtract 573; and generally,

508. To subtract a number is the same as to add its arithmetical complement.

Thus, from 5648, subtract 342.

$$\begin{array}{r} 5648 \\ -\ 342 \\ \hline 5306 \end{array} \quad \ldots \quad \ldots \quad \text{ar. com.} \begin{array}{r} 5648 \\ \bar{1}658 \\ \hline 5306 \end{array}$$

The rule given above for finding the arithmetical complement may also be expressed thus—

Subtract the given number from 1, with as many ciphers annexed as there are figures in the number, and prefix $\bar{1}$ to the remainder. When logarithms are to be subtracted, it is often, but not always, more convenient to add their arithmetical complements.

The properties of common logarithms noticed in articles (502) and (504) are peculiar to this system, in consequence of the base of the system of logarithms and that of numeration being the same; a circumstance that very much simplifies logarithmic calculations.

509. The logarithms of numbers of the natural series 1, 2, 3, 4, ... may be calculated to any extent by substituting for n in the preceding series [3] the numbers 1, 2, 3, 4, ... the results would be the logarithms of 2, 3, 4, 5, ... ; or by first finding their natural logarithms by [4], and then their common logarithms by the method in (499). It is preferable, however, to begin with some large number, as 10000, and to calculate the succeeding numbers to 100000, or any required extent; for the series will be very convergent for these high numbers; and besides the logarithms of the inferior numbers are easily found from these (504). For example, the fractional part of the logarithm of 124·56 is the same as that of the logarithm of 12456; and so on for the other inferior numbers.

510. The series [3] may be expressed in a different form. Let $L(n+1) - Ln = D$, then the formula becomes

$$D = 2M\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \ldots\};$$

when n is 100 or upwards, the second term is $= \cdot 00000004$, and the first will give the value of D correct to the 7th decimal place,

$$D = \frac{2M}{201} = \frac{\cdot 86858896}{201} = \cdot 00432134,$$

and $\qquad L(100+1) = L100 + D$, or $L101 = 2\cdot 00432134$.

This logarithm, however, is not quite correct in the 8th decimal place, for if the second term of the series be added, then $D = \cdot 00432138$, and omitting the last figure

$$L101 = 2\cdot 0043214.$$

When n is 1200 or greater, $2n$ may be taken instead of $2n+1$, and then

$$D = \frac{M}{n}.$$

Thus, for $n = 10000$, $D = \cdot 000043427$,

and hence $\qquad L10001 = L10000 + D = 4\cdot 000043427$.

So $\qquad L10002 = L10001 + D = 4\cdot 000086854$.

In a similar manner, the logarithms of 10003, 10004, 10005, ... may be calculated. If the logarithms were required to be correct only to 7 decimal places, then the value of D for $n = 10000$ is $\cdot 0000434$, and D would have the same value for several successive numbers, so that by merely adding D to $L10001$, the sum is $L10002$; and adding D to the latter, the sum is $L10003$, and so on.

When the number is still higher, as $n = 99840$, then the value of D is $= \cdot 000004349$, and for the value of $n = 99860$, D has exactly the same value, so that for all numbers between these two, D has this constant value, even to the 9th decimal place; and therefore when $L99840$ is known, the logarithm of each of the successive numbers up to 99860 is found, by merely adding this value of D to the logarithm of the preceding number. If the logarithms be limited to a smaller number of places, the difference will be constant over a greater interval. Thus, when the logarithms are extended to only seven places, the difference D is $= 44$ for all the successive numbers between 97900 and 99800.

When the logarithms of prime numbers are computed, those of composite numbers are found, by adding the logarithms of their component factors (THEO. I.)

EXAMPLE.

Given the logarithms of 2 and 7, find that of 14.

$$L2 = 0{\cdot}3010300$$
$$L7 = 0{\cdot}8450980$$

hence $$L14 = 1{\cdot}1461280$$

So the log. of $4 = L2^2 = 2L2$; $L8 = L2^3 = 3L2$.

EXERCISES.

1. Given the logarithm of $2 = 0{\cdot}3010300$ and that of $3 = 0{\cdot}4771213$, find those of 4, 6, 8, 9, and 12.

$L4 = 0{\cdot}6020600$, $L6 = 0{\cdot}7781513$, $L8 = 0{\cdot}9030900$, $L9 = 0{\cdot}9542425$, $L12 = 1{\cdot}0791812$.

2. Calculate by the formula (3) in (596) the common logarithm of 11, 13, 17, and 31, . . $L11 = 1{\cdot}0413927$, $L13 = 1{\cdot}1139434$, $L17 = 1{\cdot}2304489$, $L31 = 1{\cdot}4913617$.

SERIES.

DIFFERENTIAL METHOD.

511. The uses of the differential method are to find the successive differences of the terms of a series, or any term of the series, or the sum of a finite number of its terms.

The successive differences are divided into distinct *orders*. The differences of the *first* order are the differences of the terms of the series; those of the *second* order are the differences of the terms of the first order; those of the *third* are the differences of the terms of the second order; and so on. The differences in every case are found by subtracting each term from its succeeding term.

Thus, if the series be

1, 8, 27, 64, 125, 216, ...

then 7, 19, 37, 61, 91, ... is the 1st order of differences,

12, 18, 24, 30, ... 2d ... ... ,

and 6, 6, 6, ... 3d ... ...

PROBLEMS.

512. I. To find the first term of any order of differences.

Let the series be $a, b, c, d, e, \dots$

then the orders of differences are,

$$\begin{array}{lll}
\text{1st order} & = & b - a,\ c - b,\ d - c,\ e - d,\ \dots \\
\text{2d} \quad \dots & = & c - 2b + a,\ d - 2c + b,\ e - 2d + c,\ \dots \\
\text{3d} \quad \dots & = & d - 3c + 3b - a,\ e - 3d + 3c - b,\ \dots \\
\text{4th} \quad \dots & = & e - 4d + 6c - 4b + a,\ \dots \\
\dots & & \dots\dots \qquad \dots\dots
\end{array}$$

The coefficients of these orders of differences are evidently, for the

$$\begin{array}{ll}
\text{1st order} & 1 - 1,\ 1 - 1,\ 1 - 1,\ \dots \\
\text{2d} \quad \dots & \left\{\begin{array}{l} 1 - 1 \\ \quad - 1 + 1 \end{array}\right\} \text{or } 1 - 2 + 1, \\
\text{3d} \quad \dots & \left\{\begin{array}{l} 1 - 2 + 1 \\ \quad - 1 + 2 - 1 \end{array}\right\} \text{or } 1 - 3 + 3 - 1, \\
\text{4th} \quad \dots & \left\{\begin{array}{l} 1 - 3 + 3 - 1 \\ \quad - 1 + 3 - 3 + 1 \end{array}\right\} \text{or } 1 - 4 + 6 - 4 + 1, \\
\dots & \dots\dots \qquad\qquad \dots\dots
\end{array}$$

and so on; being plainly the terms of the successive powers of $1 - 1$, or the coefficients of the terms of the powers of a binomial $x - y$; and hence the coefficients of the nth order are

$$1 - n + \frac{n(n-1)}{2} - \frac{n(n-1)(n-2)}{2 \times 3} + \dots \quad \mp n \pm 1.$$

The last term is $+ 1$ when n is an even number, and $- 1$ when it is odd. It is also evident that the last term is the coefficient of a in the first of the nth order of differences; the last term but one is the coefficient of b; the last but two the coefficient of c, and so on; and these coefficients are the same in reverse order as in the direct order (476); therefore if d_n represent the first difference of the nth order, and $d_1, d_2, d_3, \dots$ those of the first, second, third, ... orders, then, when n is an even number,

$$d_n = a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)(n-2)}{2 \times 3}d +, \dots$$

and when n is an odd number,

$$d_n = -a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)(n-2)}{2 \times 3}d + \ldots$$

513. It is evident from the coefficients, that when $n = 1$, the value of d_n has only two terms, for then $n - 1 = 0$; when $n = 2$, this value has only three terms, for then $n - 2 = 0$; and so on.

EXAMPLES.

1. Find the first term of the second order of differences of the series $1^2, 2^2, 3^2, 4^2, \ldots$ or 1, 4, 9, 16, 25 ...

Here $n = 2$; hence take three terms of the first value of d_n, and take $a = 1$, $b = 4$, and $c = 9$; therefore

$$d_2 = a - nb + \frac{n(n-1)}{2}c = 1 - 2 \times 4 + \frac{2 \times 1}{2} \times 9$$
$$= 1 - 8 + 9 = 2.$$

The accuracy of the result may be proved by finding the differences by subtraction; thus,

1, 4, 9, 16, 25, ...
3, 5, 7, 9, ... 1st order,
2, 2, 2, ... 2d ... ;

and hence 2 is the first difference of the second order.

2. Find the first term of the fourth order of differences of the series $1^3, 2^3, 3^3, 4^3, 5^3, \ldots$ or 1, 8, 27, 64, 125 ...

Here $n = 4$; hence take five terms of the value of d_n, and $a = 1$, $b = 8$, $c = 27$, $d = 64$, $e = 125$, and hence $d_4 =$

$$1 - 4 \times 8 + \frac{4 \times 3}{2} \times 27 - \frac{4 \times 3 \times 2}{2 \times 3} \times 64 + \frac{4 \times 3 \times 2 \times 1}{2 \times 3 \times 4} \times 125$$
$$= 1 - 32 + 162 - 256 + 125 = 0,$$

or the required difference is $= 0$.

EXERCISES.

1. Find the first term of the second order of differences of the series 1, 3, 6, 10, 15, 21, ... $= 1$.

2. Find the first term of the third order of differences of the series 1, 6, 20, 50, 105, 196, ... $= 7$.

3. Find the first term of the third order of differences of the series 1, 5, 15, 35, 70, ... $= 4$.

4. Find the first term of the eighth order of differences of the series 1, 3, 9, 27, 81, ... = 256.

514. II. To find any term of a series.

Let d_1, d_2, d_3, ... represent, as above, the first terms of the different orders of differences; then by the preceding formula (512),

$$\begin{aligned} b &= a + d_1, \\ c &= \quad -a \;+ 2b + d_2, \\ d &= a - 3b + 3c + d_3, \\ e &= \quad -a \;+ 4b - 6c + 4d + d_4, \\ &\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots \end{aligned}$$

and so on to any number of terms. Hence by substituting the values of b, c, d, ...

$$\begin{aligned} b &= a + d_1, \\ c &= a + 2d_1 + d_2, \\ d &= a + 3d_1 + 3d_2 + d_3, \\ e &= a + 4d_1 + 6d_2 + 4d_3 + d_4, \\ &\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots \end{aligned}$$

and so on. It is evident from inspection that the coefficients of the different orders of differences in the value of any of the terms, as of e the fifth term, are the coefficients of the terms of a binomial involved to a power, whose exponent is one less than the number denoting the place of the terms; and hence the nth term is =

$$a + (n-1)d_1 + \frac{(n-1)(n-2)}{2}d_2 + \frac{(n-1)(n-2)(n-3)}{2 \times 3}d_3 + \ldots$$

EXAMPLES.

1. Find the 12th term of the series 2, 6, 12, 20, 30, ...

$$\begin{array}{llll} 2,\ 6,\ 12,\ 20,\ 30, & & & \\ \quad 4,\ 6,\ 8,\ 10,\ \ldots & \text{hence} & d_1 = 4, & \\ \quad\quad 2,\ 2,\ 2,\ \ldots & \ldots & d_2 = 2, & \\ \quad\quad\quad 0,\ 0,\ \ldots & \ldots & d_3 = 0, & \end{array}$$

and the succeeding orders of differences are also evidently nothing; hence the 12th term

$$= a + (n-1)d_1 + \frac{(n-1)(n-2)}{2}d_2 =$$

$$2 + 11 \times 4 + \frac{11 \times 10}{2} \times 2 = 2 + 44 + 110 = 156.$$

2. Find the nth term of the series 1, 3, 6, 10, 15, 21, ...

$$\begin{array}{ll} 1, 3, 6, 10, 15, \ldots & \\ 2, 3, 4, 5, \ldots \text{ and } & d_1 = 2, \\ 1, 1, 1, \ldots \quad \ldots & d_2 = 1, \\ 0, 0, \ldots \quad \ldots & d_3 = 0, \end{array}$$

hence nth term $= a + (n-1) \times 2 + \dfrac{(n-1)(n-2)}{2} \times 1 =$

$$1 + 2n - 2 + \frac{n^2}{2} - \frac{3n}{2} + 1 = \frac{n}{2} + \frac{n^2}{2} = \frac{1}{2}n(n+1).$$

$$\begin{array}{lll} \text{If } n = 5, & \text{the term} & = 15, \\ n = 6, & \ldots & = 21, \\ n = 20, & \ldots & = 210. \end{array}$$

EXERCISES.

1. Find the 15th term and the nth term of the series $1, 2^2, 3^2, 4^2$, ... or 1, 4, 9, 16, ... $\begin{cases} 15\text{th} = 225. \\ n\text{th} = n^2. \end{cases}$

2. Find the 20th term of the series $1, 2^3, 3^3, 4^3$, ... or 1, 8, 27, 64, 125, ... $= 8000$.

3. Find the nth term of the series 2, 6, 12, 20, 30, ... $= n(n+1)$.

515. III. To find the sum of n terms of a series.

Assume the series $0, a, a+b, a+b+c, a+b+c+d, \ldots$ whence it is evident that the $(n+1)$th term of this series is exactly the sum of n terms of the series $a, b, c, d \ldots$ The first order of differences of the former series is also $a, b, c, d, \ldots$ and the second, third, &c. orders of differences are just the first, second, third, &c. orders of differences of the series $a, b, c, d \ldots$ Hence if the same values, as in art. (512), be used for $d_1, d_2, d_3, \ldots$ the first terms of the different orders of differences of the assumed series are $a, d_1, d_2, d_3, \ldots$ and its $(n+1)$th term therefore is (514)

$$S = na + \frac{n(n-1)}{2}d_1 + \frac{n(n-1)(n-2)}{2 \times 3}d_2 + \ldots;$$

whence S is also the sum of n terms of the series $a, b, c, d \ldots$

EXAMPLES.

1. Find the sum of n terms of the odd numbers 1, 3, 5, 7, 9, ...

$$1,\ 3,\ 5,\ 7,\ 9,\ \ldots$$
$$2,\ 2,\ 2,\ 2,\ \ldots\ d_1 = 2,$$
$$0,\ 0,\ 0,\ \ldots\ d_2 = 0,$$

and

$$S = na + \frac{n(n-1)}{2}d_1 = n \times 1 + \frac{n(n-1)}{2} \times 2 = n + n^2 - n = n^2.$$

This result is a singular property of this series, from which it appears that the sum of three terms is 3^2 or 9; of four terms, 4^2 or 16; of five terms, 5^2 or 25; and so on.

2. Find the sum of n terms of the series 1, 2^2, 3^2, 4^2, 5^2 ...

It is easily found that $d_1 = 3$, $d_2 = 2$, $d_3 = 0$, and a is $= 1$; hence

$$S = na + \frac{n(n-1)}{2}d_1 + \frac{n(n-1)(n-2)}{2 \times 3}d_2$$
$$= n + \frac{3}{2}n(n-1) + \frac{1}{3}n(n-1)(n-2) = \frac{n(n+1)(2n+1)}{2 \times 3}.$$

EXERCISES.

1. Find the sum of n terms of the series 1, 2^3, 3^3, 4^3, ...

$$S = \frac{1}{4}n^2(n+1)^2, \text{ or } \left\{\frac{n(n+1)}{2}\right\}^2.$$

2. ... ... series 1, 3, 6, 10, 15, 21, ...

$$S = \frac{n(n+1)(n+2)}{2 \times 3}.$$

3. ... ... series 1, 4, 10, 20, 35, ...

$$S = \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4}.$$

4. ... ... series 1, 2^4, 3^4, 4^4, 5^4, ...

$$S = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

INTERPOLATION OF SERIES.

516. When of several equidistant terms of a series all but one are given, the latter may be found, provided that order of differences which is denoted by the same number as the number of given terms, be either nothing, or so small that it may be neglected. When the value of the required term can be found only approximately to a certain number of decimal places, it is evident that any order of differences which contains only ciphers to one further place of decimals may be neglected.

EXAMPLE.

Given the logarithms of 101, 102, 104, and 105, to find the logarithm of 103.

Here the number of terms concerned is five, or a, b, c, d, e, of which the third or c is required. Hence $n = 4$ in d_n, and as the difference of the fourth order may be neglected, the logarithms being extended to 7 places, $d_4 = 0$, or (512)

$$d_4 = a - 4b + 6c - 4d + e = 0,$$

$$\therefore\ c = \frac{1}{6}(-a + 4b + 4d - e):$$

also

$$\begin{aligned} -a &= -\log.\ 101 = -2{\cdot}0043214 \\ 4b &= 4\log.\ 102 = 8{\cdot}0344008 \\ 4d &= 4\log.\ 104 = 8{\cdot}0681332 \\ -e &= -\log.\ 105 = -2{\cdot}0211893 \\ & \qquad\qquad\quad 6)12{\cdot}0770233 \\ \therefore\ c &= \log.\ 103 = 2{\cdot}0128372 \end{aligned}$$

EXERCISES.

1. Given the logarithms of 480, 481, 482, and 484, to find that of 483.

2. Given the logarithmic sine of 5° 10′, 5° 12′, 5° 13′, and 5° 14′, to find that of 5° 11′.

The data and answers to these two exercises will be found in the logarithmic tables.

SUMMATION OF INFINITE SERIES.

517. An *infinite series* consists of an unlimited number of terms, whose values are regulated by some law.

518. The *law* of a series is a certain relation that subsists between its successive terms.

When a sufficient number of terms is given, the relation or law may frequently be found by mere inspection; thus, in the series

$$ax + \frac{a^2x^2}{b} + \frac{a^3x^3}{b^2} + \frac{a^4x^4}{b^3} + \dots$$

the law is evident, any term being found by multiplying the preceding one by $\frac{ax}{b}$.

519. The *sum* of an infinite series is a quantity to which the sum of any number of its terms, reckoning from the beginning of the series, always approaches nearer and nearer according as a greater number of terms is taken, and from which this sum differs by less than any assignable quantity. The sum of a series is therefore a *limit* to the sum of any number of its terms.

520. A *converging* series is one whose sum is finite, and such that the greater the number of terms of it that are taken, the nearer is their sum to that of the series.

521. The terms of a converging series successively diminish; but every series whose terms diminish is not a converging one.

Thus, the sum of the series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

consisting of the reciprocals of the natural numbers, is known to be infinite.

522. In a *diverging* series the terms continually increase, and are alternately plus and minus.

523. The sum of any number of terms of such a series, beginning with the first, deviates further from the value of the series, according as a greater number of its terms are taken, but if a dividend and a divisor can be found which will give the series for quotients, the true value will be found at any term by adding or subtracting the remainder.

524. A *neutral* series is one whose terms are all equal, but alternately positive and negative.

The sums of any successive numbers of terms of such a series

are alternately greater and less than the value of the series, and differ equally from it.

Thus, the series found by dividing 1 by $2 + 1$, or

$$\frac{1}{2+1} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots \quad = \frac{1}{3},$$

is a converging series.

The same quantity, or $\frac{1}{1+2}$ differently expressed, gives

$$\frac{1}{1+2} = 1 - 2 + 4 - 8 + 16 - \dots \quad = \frac{1}{3},$$

which is a diverging series.

So $$\frac{1}{1+1} = 1 - 1 + 1 - 1 + 1 - \dots \quad = \frac{1}{2},$$

which is a neutral series.

525. An *ascending* series is that in which the powers of the leading quantity continually increase; and in a *descending* series the powers of the leading quantity continually diminish.

Thus, $a + bx + cx^2 + dx^3 + \dots$ is ascending; and

$a + bx^{-1} + cx^{-2} + dx^{-3} +, \dots$ or $a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} +, \dots$ descending.

526. The *supplement* of a series is the difference between its sum and the sum of any number of its terms.

527. A *summable* series is one whose exact sum can be determined.

528. The *general* term of the following series is a function of x, in which x denotes the place of the term, reckoning from the beginning of the series; and from this term all those of the series may be derived, by giving x the successive values 1, 2, 3, 4 ...

Thus, if $\frac{a}{x}$ be the general term of a series, that series is

$$\frac{a}{1} + \frac{a}{2} + \frac{a}{3} + \frac{a}{4} + \dots$$

So if $2(x + 1)$ is the general term, the series is

$$2 . 2 + 2 . 3 + 2 . 4 + 2 . 5 + \dots$$

The following series are convergent, and their successive terms, therefore, gradually diminish and approach nearer and nearer to $\frac{1}{\infty}$ or 0, which is a limit to the magnitude of the terms.

PROBLEMS.

529. I. To find the sum of a series, whose general term is $\frac{1}{x(x+a)}$.

Let a series or the sum of a series be denoted by Σ placed before the general term, then $\Sigma\frac{1}{x}$ denotes the sum of the series whose general term is $\frac{1}{x}$, or it denotes the series itself; and from this term all the terms of the series may be formed by substituting as values of x, successively, the terms of the series of natural numbers; thus,

$$\Sigma\frac{1}{x} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots \frac{1}{x} + \dots$$

So
$$\Sigma\frac{1}{x+4} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots\dots \frac{1}{x+4} + \dots$$

Similarly, the series whose general term is $\frac{1}{x+a}$, is

$$\Sigma\frac{1}{x+a} = \frac{1}{a+1} + \frac{1}{a+2} + \frac{1}{a+3} + \dots \frac{1}{x+a} + \dots$$

The series $\Sigma\frac{1}{x}$ may also be written thus:

$$\Sigma\frac{1}{x} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+2} + \dots \frac{1}{x} + \dots$$

Now if, according to the common method of summation by subtraction, the differences between the corresponding terms of these two series be taken, these differences will form the series (136),

$$\frac{a}{1(a+1)} + \frac{a}{2(a+2)} + \frac{a}{3(a+3)} + \dots \frac{a}{x(x+a)}.$$

This series is denoted by $\Sigma\frac{a}{x(x+a)}$, or $a\Sigma\frac{1}{x(x+a)}$; but the series $\Sigma\frac{1}{x+a}$ is evidently the same as the series $\Sigma\frac{1}{x}$ after its first

a terms; therefore

$$a\Sigma\frac{1}{x(x+a)} = \Sigma\frac{1}{x} - \Sigma\frac{1}{x+a},$$

the difference between the two latter series being exactly the first a terms of the series $\Sigma\frac{1}{x}$.

Hence let $S =$ the sum of a terms of the series $\Sigma\frac{1}{x}$, after the operation of summing has been effected, then

$$\Sigma\frac{a}{x(x+a)} = S, \text{ or } a\Sigma\frac{1}{x(x+a)} = S, \text{ or } \Sigma\frac{1}{x(x+a)} = \frac{S}{a}.$$

That is, to obtain the sum of the series $\Sigma\frac{1}{x(x+a)}$, divide the sum of the first a terms of the series $\Sigma\frac{1}{x}$ by a, and the quotient will be the required result.

530. The sum of the series $\Sigma\frac{1}{x(x+a)}$ may also be found by actually writing down a sufficient number of terms of the two series, $\Sigma\frac{1}{x}$ and $\Sigma\frac{1}{x+a}$, and taking the difference of their corresponding terms, which will produce the series $\Sigma\frac{a}{x(x+a)}$, and then observing that the difference between these two series is also the first a terms of $\Sigma\frac{1}{x}$, and hence this latter quantity is $=$ $\Sigma\frac{a}{x(x+a)}$, which being divided by a, the quotient is $= \Sigma\frac{1}{x(x+a)}$.

EXAMPLES.

1. Find the sum of the series $\frac{1}{1\times 3}, \frac{1}{2\times 4}, \frac{1}{3\times 5}, \ldots\ldots \frac{1}{x(x+2)}$, or $\Sigma\frac{1}{x(x+2)}$.

Here $a = 2$, and the sum of two terms of the series $\Sigma\frac{1}{x}$ is $S = \frac{1}{1} + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$;

hence $$a\Sigma\frac{1}{x(x+2)} = 2\Sigma\frac{1}{x(x+2)} = S = \frac{3}{2},$$

or $$\Sigma\frac{1}{x(x+2)} = \frac{1}{2}\times\frac{3}{2} = \frac{3}{4}.$$

The sum may also be found by the common method of subtraction (530); thus,

$$\Sigma\frac{1}{x} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

$$\Sigma\frac{1}{x+2} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$$

The excess of the former series above the latter is evidently $1 + \frac{1}{2} = \frac{3}{2}$; and the difference between the corresponding terms gives the series—

$$\frac{2}{1.3} + \frac{2}{2.4} + \frac{2}{3.5} + \frac{2}{4.6} + \dots = \frac{3}{2}.$$

Hence $$\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \frac{1}{4.6} + \dots = \frac{3}{4}.$$

531. To find the sum of n terms of the series $\Sigma\frac{1}{x(x+a)}$, take the sum of the first a terms of the series $\Sigma\frac{1}{x}$, and the sum of the last a terms of the first n terms of the series $\Sigma\frac{1}{x+a}$; divide the excess of the former sum above the latter by a; and the quotient will be the quantity required.

For if the first n terms of the two series $\Sigma\frac{1}{x}$ and $\Sigma\frac{1}{x+a}$ be written down, it is evident, by giving a any numerical value, that the two series of terms are identical, except the first a terms of the former and the last a terms of the latter; and therefore the difference between these two series carried to n terms is = the excess of the first a terms of the former above the last a terms of the latter; but the difference between the corresponding terms of these two series is = the first n terms of the series $\Sigma\frac{a}{x(x+a)}$, or $a\Sigma\frac{1}{x(x+a)}$; hence the former difference divided by a is the result required.

2. To find the sum of n terms of the same series (ex. 1).

The first a terms of $\Sigma\frac{1}{x}$ are $1+\frac{1}{2}=\frac{3}{2}$, for $a=2$; and the sum of the last a terms of the first n terms of $\Sigma\frac{1}{x+2}$, that is, of the last two terms, is $=\frac{1}{n+1}+\frac{1}{n+2}=\frac{2n+3}{(n+1)(n+2)}$; and hence the sum required is

$$=\frac{1}{2}\left(\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}\right)=\frac{n(3n+5)}{4(n+1)(n+2)}.$$

532. This sum may be found by subtraction also, thus:—

$$\Sigma\frac{1}{x} \text{ to } n \text{ terms} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots\frac{1}{n-1}+\frac{1}{n}$$

$$\Sigma\frac{1}{x+2} \quad \dots \quad = \frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}.$$

Now, the difference between these series is evidently $1+\frac{1}{2}-\left(\frac{1}{n+1}+\frac{1}{n+2}\right)=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}$; and the difference between the corresponding terms is

$$\frac{2}{1.3}+\frac{2}{2.4}+\frac{2}{3.5}+\dots+\frac{2}{n(n+2)};$$

hence $$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\dots+\frac{1}{n(n+2)}$$

$$=\frac{1}{2}\left(\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}\right)=\frac{n(3n+5)}{4(n+1)(n+2)}.$$

The student may perform the following exercises, both by the general formula, and by the method of subtraction exemplified above (530), which may be easily effected by finding the differences of the first terms, the second terms, and so on of the series $\Sigma\frac{1}{x}$ and $\Sigma\frac{1}{x+a}$.

EXERCISES.

1. Find the sum of the series $\Sigma \frac{1}{x(x+1)}$, and the sum of n terms of it, ... $S = 1$, sum of n terms $= \frac{n}{n+1}$.

2. Find the sum of the series $\Sigma \frac{1}{x(x+3)}$, and the sum of n terms of it, . . $S = \frac{11}{18}$, sum of n terms $= \frac{n(11n^2 + 48n + 49)}{18(n+1)(n+2)(n+3)}$.

3. Find the sum of the series $\Sigma \frac{1}{x(x+4)}$, . . $S = \frac{25}{48}$.

533. II. To find the sum of a series whose general term is $\frac{1}{(mx+a)(mx+b)}$.

Since $$\frac{1}{mx+a} - \frac{1}{mx+b} = \frac{b-a}{(mx+a)(mx+b)},$$

each term of the given series, multiplied by $(b - a)$, is equal to the difference between the corresponding terms of the series whose general terms are $\frac{1}{mx+a}$ and $\frac{1}{mx+b}$. And in order that the terms of the latter series may coincide with those of the former, after a certain number of terms, some value of x, as x', must make $mx + a$ of the same value as $mx + b$ when $x = 1$; that is,

$$mx' + a = m + b, \text{ therefore } x' = 1 + \frac{b-a}{m}.$$

Hence that x' may be a whole number, $b - a$ must be a multiple of m; therefore $\frac{b-a}{m} = r$, or $b - a = rm$; that is, $b = a + rm$; hence the given series must be of the form $\frac{1}{(mx+a)(mx+a+rm)}$,

or $$\frac{1}{(mx+a)\{m(x+r)+a\}},$$

and $$rm\,\Sigma \frac{1}{(mx+a)\{m(x+r)+a\}} = S,$$

or $$\Sigma \frac{1}{(mx+a)\{m(x+r)+a\}} = \frac{S}{rm},$$

where S is the sum of r terms of the series $\Sigma\frac{1}{mx+a}$; for this series will evidently exceed the series $\Sigma\frac{1}{m(x+r)+a}$ by its first r terms.

EXAMPLE.

Find the sum of the series whose general term is $\frac{1}{(3x+2)(3x+8)}$.

Here $a=2$, $b=8$, $m=3$, $rm=b-a=6$; hence $r=\frac{6}{3}=2$.

Hence $S=\frac{1}{3+2}+\frac{1}{6+2}=\frac{1}{5}+\frac{1}{8}=\frac{13}{40}$ and $\frac{S}{rm}=\frac{13}{240}=$ sum required.

Or the sum may be found thus (530),

$$\Sigma\frac{1}{3x+2}=\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}+\ldots$$

$$\Sigma\frac{1}{3x+8}=\frac{1}{11}+\frac{1}{14}+\frac{1}{17}+,\ldots$$

and the difference is evidently $\frac{1}{5}+\frac{1}{8}=\frac{13}{40}$; and taking also the difference of the corresponding terms, it gives

$$\frac{6}{5.11}+\frac{6}{8.14}+\frac{6}{11.17}+\ldots=\frac{13}{40};$$

hence $$\frac{1}{5.11}+\frac{1}{8.14}+\frac{1}{11.17}+\ldots=\frac{1}{6}\cdot\frac{13}{40}=\frac{13}{240}.$$

EXERCISES.

1. Find the sum of the series $\Sigma\frac{1}{(x+3)(x+5)}$, . $S=\frac{9}{40}$.

2. ... ... $\Sigma\frac{1}{(2x+3)(2x+7)}$, . $S=\frac{3}{35}$.

3. ... ... $\Sigma\frac{1}{(4x-3)(4x+1)}$, . $S=\frac{1}{4}$.

534. If n terms of the series $\Sigma\frac{1}{mx+a}$ be taken, and also n terms

of the series $\Sigma\frac{1}{m(x+r)+a}$, it is evident that the last $n-r$ terms of the former will be identical with the first $n-r$ terms of the latter; and therefore

The difference between the whole n terms of the two series will be equal to the excess of the first r terms of the former above the last r terms of the latter.

But the difference between the corresponding terms of the n terms of the two series is also equal to n terms of the series

$$\Sigma\frac{rm}{(mx+a)\{m(x+r)+a\}}, \text{ or } rm\ \Sigma\frac{1}{(mx+a)\{m(x+r)+a\}}.$$

This will appear evident by writing n terms of these two series,

$$\frac{1}{m+a}+\frac{1}{2m+a}+\frac{1}{3m+a}+\dots\frac{1}{(r-1)m+a}+\frac{1}{rm+a}$$
$$+\{\frac{1}{(r+1)m+a}+\dots\frac{1}{nm+a}\},$$

and
$$\{\frac{1}{(r+1)m+a}+\frac{1}{(r+2)m+a}+\dots\frac{1}{nm+a}\}$$
$$+\frac{1}{(n+1)m+a}+\dots\frac{1}{(n+r)m+a}.$$

The identical terms are enclosed in braces, and the difference between these n terms of the two series is evidently the excess of the first r terms of the former above the last r terms of the latter, and the difference between the corresponding terms gives the first n terms of the series

$$rm\Sigma\frac{1}{(mx+a)\{m(x+r)+a\}}.$$

When $r=1$, n terms of the last series is therefore

$$=\left(\frac{1}{m+a}-\frac{1}{(n+1)m+a}\right)\frac{1}{m}=\frac{n}{(m+a)\{(n+1)m+a\}},$$

and the sum of n terms for any other value of r may be similarly found.

Thus, the sum of n terms of the series in the first exercise given above, or $\Sigma\frac{1}{(x+3)(x+5)}$, for which $m=1$, $a=3$, $b=5$,

$rm = b - a = 2$, and therefore $r = 2$, is

$$= \left(\frac{1}{m+a} + \frac{1}{2m+a} - \frac{1}{(n+1)m+a} - \frac{1}{(n+2)m+a}\right)\frac{1}{rm}$$

$$= \left(\frac{1}{4} + \frac{1}{5} - \frac{1}{n+4} - \frac{1}{n+5}\right) \times \frac{1}{2} = \frac{9n^2 + 41n}{40(n+4)(n+5)}.$$

535. III. To find the sum of the series whose general term is

$$\frac{1}{(mx+a)(mx+b)(mx+c)}.$$

Since
$$\frac{1}{(mx+a)(mx+b)} - \frac{1}{(mx+b)(mx+c)}$$
$$= \frac{c-a}{(mx+a)(mx+b)(mx+c)};$$

$$\therefore (c-a)\Sigma\frac{1}{(mx+a)(mx+b)(mx+c)}$$
$$= \Sigma\frac{1}{(mx+a)(mx+b)} - \Sigma\frac{1}{(mx+b)(mx+c)}.$$

By the former problem (533), the sums of the two latter series in the second member can be found, provided that $(b - a)$ and $(c - b)$ are both multiples of m. The differences of these sums being divided by $(c - a)$ will give the sum of the proposed series.

536. When $(b - a)$ and $(c - b)$ are different multiples of m, the sum of the proposed series may be found by the preceding method.

537. When $(b - a)$ and $(c - b)$ are the same multiple of m, the sum may be more readily found; for then the same value of x, as x', will make the factors in the denominator of some term of the first series equal to those in the first term of the second series, or if $b - a = c - b = mr$,

and $$mx' + a = m + b, \text{ and } mx' + b = m + c;$$

then $$x' = 1 + r;$$

whence the series $\Sigma\frac{1}{(mx+a)(mx+b)}$ after its first r terms

coincides with the series $\Sigma\frac{1}{(mx+b)(mx+c)}$.

Hence

$$2rm\Sigma\frac{1}{(mx+a)\{m(x+r)+a\}\{m(x+2r)+a\}} = S;$$

where S is the sum of the first r terms of the series

$\Sigma\frac{1}{(mx+a)\{m(x+r)+a\}}$, and the sum of the given series is then $=\frac{S}{2rm}$.

EXAMPLE.

Find the sum of the series $\Sigma\frac{1}{(2x+3)(2x+7)(2x+11)}$.

Here $m=2$, $a=3$, $mr=7-3=4$; therefore $r=2$, and $2rm=8$,

also $$S=\frac{1}{5\times 9}+\frac{1}{7\times 11};$$

hence sum $$=\frac{S}{2rm}=\frac{1}{8}\left(\frac{1}{45}+\frac{1}{77}\right)=\frac{61}{13860}.$$

EXERCISES.

1. Find the sum of the series $\Sigma\frac{1}{(2x+1)(2x+5)(2x+9)}$, $S=\frac{11}{1260}$.

2. ... ... $\Sigma\frac{1}{x(x+1)(x+2)}$, . . $S=\frac{1}{4}$.

538. IV. To find the sum of the series whose general term is

$$\frac{1}{(mx+a)(mx+b)(mx+c)(mx+d)}.$$

Since $$\frac{1}{(mx+a)(mx+b)}-\frac{1}{(mx+c)(mx+d)}$$
$$=\frac{m(c+d)x+cd-m(a+b)x-ab}{(mx+a)(mx+b)(mx+c)(mx+d)}$$

in order to make x disappear from the numerator of the second member, assume $c+d=a+b$, and then the difference between the series whose general terms compose the first member being divided by $(cd-ab)$, the quotient would be the sum of the proposed series when $a+b=c+d$.

When $(b-a)$, $(c-b)$, and $(d-c)$, are the same multiples of m, the sum may be more readily found thus—

$$\frac{1}{(mx+a)(mx+b)(mx+c)}-\frac{1}{(mx+b)(mx+c)(mx+d)}$$
$$=\frac{d-a}{(mx+a)(mx+b)(mx+c)(mx+d)};$$

and if $b - a = c - b = d - c = rm$; and therefore $d - a = 3rm$, then the sum of the proposed series is

$$\Sigma \frac{1}{(mx+a)\{m(x+r)+a\}\{m(x+2r)+a\}\{m(x+3r)+a\}} = \frac{S}{3rm},$$

where S is the sum of the first r terms of the series

$$\Sigma \frac{1}{(mx+a)(mx+b)(mx+c)},$$

where $b = a + mr$, and $c = a + 2mr$.

This is proved exactly as the similar formula (537) in the preceding problem.

EXAMPLE.

Find the sum of the series whose general term is

$$\frac{1}{(2x+1)(2x+3)(2x+5)(2x+7)}.$$

Here $a = 1$, $b = 3$, $m = 2$, $rm = b - a = 2$; therefore $r = 1$, and $3rm = 6$;

$$\text{hence } S = \frac{1}{3.5.7}, \text{ and } \frac{S}{3rm} = \frac{1}{6} \times \frac{1}{105} = \frac{1}{630}.$$

Or by actually subtracting the two series,

$$\frac{1}{3.5.7} + \frac{1}{5.7.9} + \frac{1}{7.9.11} + \ldots$$

$$\frac{1}{5.7.9} + \frac{1}{7.9.11} + \frac{1}{9.11.13} +, \ldots$$

$$\text{then } \frac{6}{3.5.7.9} + \frac{6}{5.7.9.11} + \frac{6}{7.9.11.13} + \ldots = \frac{1}{3.5.7},$$

and $\frac{1}{6}$ of this gives the sum of the proposed series

$$= \frac{1}{6} \times \frac{1}{3.5.7} = \frac{1}{630}.$$

EXERCISES.

1. Find the sum of the series $\Sigma \frac{1}{(x+2)(x+3)(x+4)(x+5)}$,

$$S = \frac{1}{180}.$$

2. ... ... $\Sigma \frac{1}{x(x+1)(x+2)(x+3)}$, $S = \frac{1}{18}$.

539. It is evident that this method may be very easily extended to any series whose general term is of the form

$$\frac{1}{(mx+a)\{m(x+r)+a\}\{m(x+2r)+a\}\ldots\{m(x+n-1r)+a\}}.$$

If the terms of the series have a constant numerator, as p instead of 1, the sum will be just p times greater.

By methods somewhat similar, the sum of any series, the numerators of whose general term is of the form $px+e$, and denominator of the above form, may be found, when the denominators contain more than two factors.

540. V. To find the sum of the series whose general term is

$$\frac{px+e}{(mx+a)\{m(x+r)+a\}\{m(x+2r)+a\}}.$$

Since

$$\frac{m(px+e)}{(mx+a)\{m(x+r)+a\}\{m(x+2r)+a\}}=\frac{p}{(mx+a)\{m(x+r)+a\}}$$
$$-\frac{p(2mr+a)-me}{(mx+a)\{m(x+r)+a\}\{m(x+2r)+a\}}.$$

If S denote the difference between the sums of the two series whose general terms form the second member of this equation, then the sum of the proposed series is proved, as in former propositions, to be $=\frac{S}{m}$.

EXAMPLE.

Find the sum of the series whose general term is

$$\frac{x+2}{(2x+1)(2x+3)(2x+5)}.$$

Here $p=1$, $e=2$, $m=2$, $a=1$, $mr=3-1=2$; hence $r=1$, and $p(2mr+a)-me=1$;

$$\therefore\ \Sigma\frac{1}{(2x+1)(2x+3)}=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6},$$

and $$\Sigma\frac{1}{(2x+1)(2x+3)(2x+5)}=\frac{1}{4}\times\frac{1}{3\,.\,5}=\frac{1}{60}.$$

Hence the sum of the given series $=\frac{1}{2}\left(\frac{1}{6}-\frac{1}{60}\right)=\frac{1}{2}\times\frac{9}{60}=\frac{3}{40}.$

EXERCISE.

Find the sum of the series whose general term is

$$\frac{3x-2}{(2x-1)(2x+1)(2x+3)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad S = \frac{5}{24}.$$

541. VI. To find the sum of a series whose general term is

$$\frac{px+e}{(mx+a)\{m(x+r)+a\}\{m(x+2r)+a\}\{m(x+3r)+a\}}.$$

It may be shewn, as in the last problem, that the sum of the proposed series is the mth part of the difference between two series whose general terms are

$$\frac{p}{(mx+a)\{m(x+r)+a\}\{m(x+2r)+a\}},$$

and

$$\frac{p(3mr+a)-me}{(mx+a)\{m(x+r)+a\}\{m(x+2r)+a\}\{m(x+3r)+a\}}.$$

EXAMPLE.

Find the sum of the series whose general term is

$$\frac{2x+3}{x(x+1)(x+2)(x+3)}.$$

Here $p=2$, $e=3$, $a=0$, $m=1$, $rm=1$; hence $r=1$, and $p(3mr+a)-me=3$;

$$\therefore \Sigma\frac{2}{x(+1)(x+2)} = \frac{1}{2} \times \frac{2}{1 \times 2} = \frac{1}{2},$$

and $$\Sigma\frac{3}{x(x+1)(x+2)(x+3)} = \frac{1}{3} \times \frac{3}{1.2.3} = \frac{1}{6},$$

and the required sum $= \dfrac{1}{m}\left(\dfrac{1}{2} - \dfrac{1}{6}\right) = \dfrac{1}{3}.$

EXERCISE.

Find the sum of the series whose general term is

$$\frac{4x+6}{x(x+1)(x+2)(x+3)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad S = \frac{2}{3}.$$

542. The sum of a series, such as $\Sigma \dfrac{px^2 + qx + c}{(x+a)(x+b)(x+c)}$, may be found by a method somewhat similar. Thus, assume

$$\frac{px^2 + qx + c}{(x+a)(x+b)(x+c)} = \frac{Ax + B}{(x+a)(x+b)} - \frac{C}{(x+a)(x+b)(x+c)},$$

and by reducing the fractions in the second member to one, and equating its numerator with $px^2 + qx + c$, the values of A, B, and C, would be found by the principle of undetermined coefficients, as their values remain constant, while those of x vary; and if $b - a = c - b$, then by the preceding methods, the sums of the series of which the terms in the second member are the general terms, can be found, the difference between which would be the sum of the proposed series. This method may be easily extended.

REVERSION OF SERIES.

543. When the value of one quantity is expressed in terms of another by an ascending series, the value of the latter may also be similarly developed by the method of the *reversion*, or *inverse method* of series.

544. Let $y = ax + bx^2 + cx^3 + dx^4 + \dots$ [1],

the coefficients a, b, c, ... being known; in order to find the development of x in terms of y, assume the series

$$x = Ay + By^2 + Cy^3 + Dy^4 + \dots \qquad [2],$$

in which the coefficients A, B, C, ... are undetermined.

Find the values of y^2, y^3, y^4, ... from [1], thus:—

$$\begin{array}{ll}
y^2 = & a^2x^2 + 2abx^3 + \left.\begin{array}{l} b^2 \\ +\,2ac \end{array}\right| x^4 + \left.\begin{array}{l} 2bc \\ +\,2ad \end{array}\right| x^5 + \dots \\
y^3 = & \qquad\quad a^3x^3 + 3a^2bx^4 + 3ab^2x^5 + \dots \\
y^4 = & \qquad\qquad\qquad a^4x^4 + 4a^3bx^5 + \dots \\
y^5 = & \qquad\qquad\qquad\qquad\qquad a^5x^5 + \dots
\end{array}$$

Substituting these values in [2], and arranging

$$0 = \left.\begin{array}{r} Aa \\ -\;1 \end{array}\right| x + \left.\begin{array}{l} Ab \\ +\,Ba^2 \end{array}\right| x^2 + \left.\begin{array}{l} Ac \\ +\,2Bab \\ +\,Ca^3 \end{array}\right| x^3 + \left.\begin{array}{l} Ad \\ +\,Bb^2 \\ +\,2Bac \\ +\,3Ca^2b \\ +\,Da^4 \end{array}\right| x^4 + \dots$$

Equating with zero the coefficients of the different powers of x (459).

$$Aa - 1 = 0,\ Ab + Ba^2 = 0,\ Ac + 2Bab + Ca^3 = 0,$$
$$Ad + Bb^2 + 2Bac + 3Ca^2b + Da^4 = 0,\ \ldots$$

from which are derived

$$A = \frac{1}{a},\ B = -\frac{Ab}{a^2} = -\frac{b}{a^3},\ C = -\frac{ac - 2b^2}{a^5},\ D = -\frac{a^2d - 5abc + 5b^3}{a^7}.$$

Hence the development of x in terms of y is by [2],

$$x = \frac{1}{a}y - \frac{b}{a^3}y^2 - \frac{ac - 2b^2}{a^5}y^3 - \frac{a^2d - 5abc + 5b^3}{a^7}y^4 - \ldots\ [3].$$

545. If the given series has a constant term prefixed; thus,

$$y = \alpha + ax + bx^2 + cx^3 + dx^4 + \ldots \qquad [1],$$

then assume $y - \alpha = z$, and the expression becomes

$$z = ax + bx^2 + cx^3 + dx^4 + \ldots \qquad [2],$$

and the value of x developed in terms of z is found in the same manner as its expansion in terms of y was found before (544), by assuming

$$x = Az + Bz^2 + Cz^3 + Dz^4 + \ldots \qquad [3],$$

the coefficients of which would be found to have the same value as before; so that if z be substituted for y in [3], the result is the required development of x; and then $y - \alpha$ being substituted for z, the result is

$$x = \frac{1}{a}(y - \alpha) - \frac{b}{a^3}(y - \alpha)^2 - \frac{ac - 2b^2}{a^5}(y - \alpha)^3$$
$$- \frac{a^2d - 5abc + 5b^3}{a^7}(y - \alpha)^4 - \ldots \qquad [4].$$

546. When the given series contains the odd powers of x, assume for x another series containing the odd powers of y. Thus, let

$$y = ax + bx^3 + cx^5 + dx^7 + \ldots$$

to develop x in terms of y, assume

$$x = Ay + By^3 + Cy^5 + Dy^7 + \ldots$$

Substitute in the latter series the values of y, y^3, y^5, ... derived from the former in the same manner as was done in (544); then, having arranged the result according to the powers of x, the

coefficients of x, x^3, x^5, ... being each equated with zero, the values are found as formerly to be

$$A = \frac{1}{a},\ B = -\frac{b}{a^4},\ C = -\frac{ac - 3b^2}{a^7},\ D = -\frac{a^2d - 8abc + 12b^3}{a^{10}},\ldots$$

and hence,

$$x = \frac{1}{a}y - \frac{b}{a^4}y^3 - \frac{ac - 3b^2}{a^7}y^5 - \frac{a^2d - 8abc + 12b^3}{a^{10}}y^7 - \ldots$$

The following exercises will be easily solved by substituting for a, b, c, d, ... their values in the given series, and then finding the values of A, B, C, D, ... as in the following example:—

Let $y = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\,.\,3}x^3 + \frac{1}{2\,.\,3\,.\,4}x^4 +, \ldots$ to find the development of x in terms of y.

By [1] in art. (545), it appears that here $\alpha = 1$, $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{2\,.\,3}$, $d = \frac{1}{2\,.\,3\,.\,4}$, ... and hence the values of A, B, C, ... which are the same as in art. (544), are

$$A = \frac{1}{a} = \frac{1}{1} = 1,\ B = -\frac{Ab}{a^2} = -\frac{1 \times \frac{1}{2}}{1} = -\frac{1}{2},$$

$$C = -\frac{ac - 2b^2}{a^5} = -(1 \times \frac{1}{2.3} - 2 \times \frac{1}{4}) = -\left(\frac{1}{6} - \frac{1}{2}\right) = \frac{1}{3},$$

and the value of D is similarly found to be $D = -\frac{1}{4} \ldots$; and hence by [4] art. (545),

$$x = \frac{y-1}{1} - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \frac{(y-1)^4}{4} + \ldots$$

The given series is the value of a^x (486) when $k = 1$, or of $e^x = y$, where x is the Napierian log. of y; and hence the series found for x is just the series for the Napierian log. of y or ly in terms of y, and coincides with the series for $L'(1 + x)$ in art. (496), when y is taken in that series for $1 + x$, and consequently $y - 1$ for x, and l instead of L'; for then $L'e = le = 1$, and $L'(1 + x) = ly$.

EXERCISES.

1. Given the series $y = x - x^2 + x^3 - x^4 + \ldots$ to find the value of x in terms of y, . . . $x = y + y^2 + y^3 + y^4 + \ldots$

2. Given $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ to find x in terms of y, . . . $x = y + \frac{1}{2}y^2 + \frac{1}{2\,.\,3}y^3 + \frac{1}{2\,.\,3\,.\,4}y^4 + \ldots$

3. Given $y = x - \frac{1}{2\,.\,3}x^3 + \frac{1}{2\,.\,3\,.\,4\,.\,5}x^5 - \frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}x^7 + \ldots$ to find x in terms of y,

$$x = y + \frac{1}{2\,.\,3}y^3 + \frac{3}{2\,.\,4\,.\,5}y^5 + \frac{3\,.\,5}{2\,.\,4\,.\,6\,.\,7}y^7 + \frac{3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8\,.\,9}y^9 + \ldots$$

4. Given $x + ax^2 + bx^3 + cx^4 + \ldots = gy + hy^2 + ky^3 + ly^4 \ldots$ to find y in terms of x,

$$y = \frac{x}{g} + \frac{(ag^2 - h)x^2}{g^3} + \frac{\{bg^4 - kg - 2h(ag^2 - h)\}x^3}{g^5} + \ldots$$

GENERAL SOLUTION OF THE HIGHER EQUATIONS HAVING NUMERICAL COEFFICIENTS.

547. In order to obtain a general solution of equations of all degrees which have numerical coefficients, it will be necessary to investigate some of the properties of equations which have not been given in the previous part of the work.

Every equation involving only one unknown quantity and its powers may be reduced, by collecting its terms, transposing them all to one side, and dividing by the coefficient of the highest power of x, if it is not one, to the form

$$x^n + Ax^{n-1} + Bx^{n-2} +, \&c., Lx^2 + Mx + N = 0;$$

where the coefficients, A, B, &c., L, M, and the absolute term N are functions of the roots, and may be either positive or negative; or some of them may be zero, and the corresponding powers of x will be wanting in the equation; and the exponent n represents the highest power of x in the equation, and it is called an equation of the nth degree.

548. If several simple equations involving the same unknown

symbol be multiplied continually together, the product will be an equation of as many dimensions as there are simple equations employed.

Thus, the product of two simple equations is a quadratic; the continued product of three simple equations is a cubic; that of four, a biquadratic; and so on to any number of dimensions.

For, let x be any variable unknown quantity, and let the given quantities a, b, c, d, ... be its several values, so that $x = a$, $x = b$, $x = c$, $x = d$, &c.; these, by transposition, become $x - a = 0$, $x - b = 0$, $x - c = 0$, $x - d = 0$, &c. If the continued product of these simple equations be taken—namely $(x - a)(x - b)(x - c)(x - d)$, and so on—it will produce an equation $(= 0)$ of as many dimensions as there are factors or simple equations employed in its composition.

For example—

$$\begin{array}{rl}
\text{Let} & x - a = 0 \\
\text{be multiplied into} & x - b = 0 \\
\hline
\end{array}$$

The product is
$$\begin{array}{rr|l}
x^2 & -a & x + ab = 0, \text{ a quadratic.} \\
& -b &
\end{array}$$

Multiplied into $\quad x - c = 0$

The product is
$$\begin{array}{rr|r|l}
x^3 & -a & x^2 + ab & x - abc = 0, \text{ a cubic.} \\
& -b & +ac & \\
& -c & +bc &
\end{array}$$

Multiplied into $\quad x - d = 0$

The product is
$$\begin{array}{rr|r|r|l}
x^4 & -a & x^3 + ab & x^2 - abc & x + abcd = 0, \text{ a biquad-} \\
& -b & +ac & -abd & \text{ratic.} \\
& -c & +ad & -acd & \\
& -d & +bc & -bcd & \\
& & +bd & & \\
& & +cd & &
\end{array}$$

&c. &c.

From the inspection of these equations, remembering that a, b, c, and d, are the roots of the equation, it appears that—

549. The product of two simple equations is a quadratic.

550. The continued product of three simple equations, or of one quadratic and one simple equation, is a cubic.

551. The continued product of four simple equations, or of two quadratics, or of one cubic and one simple equation, is a biquadratic; and so on for higher equations.

552. The coefficient of the first term or highest power in each equation is unity.

553. The coefficient of the second term in each, is the sum of the roots with their signs changed.

Thus, in the quadratic, whose roots are $+ a$ and $+ b$, the coefficient is $- a - b$; in the cubic, whose roots are $+ a + b$, and $+ c$, it is $- a - b - c$; in the biquadratic, whose roots are $+ a, + b, + c, + d$, it is $- a - b - c - d$, &c.

554. The coefficient of the third term in each, is the sum of all the products that can possibly arise by combining the roots, with their signs changed, two and two.

Thus, in the cubic, the coefficient of the third term is $+ ab + ac + bc$; in the biquadratic, it is $+ ab + ac + ad + bc + bd + cd$, &c.

555. The coefficient of the fourth term in each, is the sum of all the products that can possibly arise by combining the roots, with their signs changed, three by three.

Thus, in the biquadratic, the coefficient of the fourth term is $- abc - abd - acd - bcd$.

In like manner, in higher equations, the coefficient of the fifth term will be the sum of all the products of the roots, having their signs changed, combined four by four; that of the sixth, the sum of the products of the roots, with their signs changed, combined five by five, &c.

556. The last, or absolute term, is always the continued product of all the roots, having their signs changed.

Thus, in the quadratic, whose roots are $+ a$ and $+ b$, the last term is $+ ab$ (or $- a \times - b$); in the cubic, the absolute term is $- abc$ (or $- a \times - b \times - c$); in the biquadratic, the absolute term is $+ abcd$ $(= - a \times - b \times - c \times - d)$, &c.

557. Cor. Hence every root of an equation is a divisor of its last or absolute term.

558. When the roots are all positive, the signs of the terms of the equation will be alternately positive and negative; and conversely, when the signs of the terms of the equations are alternately positive and negative, all the roots will be positive.

Cor. Hence, if the signs of the even terms be changed, the signs of all the roots of the equation will be changed.

559. Let now the roots of the equations, referred to above, be supposed negative; that is $x = - a$, $x = - b$, $x = - c$, $x = - d$, &c.; then, by transposition, $x + a = 0$, $x + b = 0$, $x + c = 0$, $x + d = 0$, &c.; the product of these, or $(x + a)(x + b)(x + c)(x + d)$, &c. will be an equation, having all its terms positive; for

since all the quantities composing the factors are $+$, it is plain that the products will all be $+$

560. Hence, when the signs of all the roots of the simple equations are $-$, the signs of all the terms of the equation compounded of them will be $+$; and conversely, when the signs of all the terms of an equation are $+$, the signs of all its roots will be $-$

EXERCISES.

1. Form the equation whose roots are $+3$, $+5$, and -2.

Here $x-3=0$, $x-5=0$, and $x+2=0$; hence the equation is $(x-3)(x-5)(x+2) = x^3 - 6x^2 - x + 30 = 0$.

2. Form the equation whose roots are $+1$, $+2$, -4, and -7,
$= x^4 + 8x^3 - 3x^2 - 62x + 56 = 0$.

3. ... ... -1, -2, $+4$, and $+7$,
$= x^4 - 8x^3 - 3x^2 + 62x + 56 = 0$.

4. ... ... $+6$, $+5$, $+4$, -3, and -1, . . $= x^5 - 11x^4 + 17x^3 + 131x^2 - 258 - 360 = 0$.

5. Form the equation whose roots are $2 + 3\sqrt{-1}$, $2 - 3\sqrt{-1}$, $+4$, -3, and -1, . . $= x^5 - 12x^2 - 169x - 156 = 0$.

6. Form the equation whose roots are $+7$, $+3$, -8, -4, and -2, . $= x^5 + 4x^4 - 63x^3 - 202x^2 + 536x + 1344 = 0$.

561. Theorem. An equation cannot have more positive roots than it has changes of sign from $+$ to $-$, or from $-$ to $+$, in the terms of its first member; nor can it have a greater number of negative roots than of permanencies, or successive repetitions of the same sign.

Let the signs of a complete equation, taken in order, be

$$+ - - + - + + + - ;$$

if this equation be multiplied by $x - a$, so as to introduce one more positive root, there will be at least one more change of sign; for in multiplying by x, the signs of the product will be the given signs; and in multiplying by $-a$, the signs will all be changed, and removed one place to the right, so that they will stand thus:

$$\begin{array}{cccccccccc} + & - & - & + & - & + & + & + & - & \\ & - & + & + & - & + & - & - & - & + \\ \hline + & - & \mp & + & - & + & \pm & \pm & - & + \end{array}$$

The number of changes in the first four signs of the product is

the same as in the original equation, whether the third sign be taken $-$ or $+$; and as the fifth and sixth signs are the same as before, the changes to the sixth term inclusive are the same as in the original equation. The number of changes of sign in the original and derived equations will therefore depend on the changes from the sixth term to the end. In the original equation there is evidently but one change; but in the derived equation, if the seventh and eighth terms be both taken with the sign $+$, or both with the sign $-$, there are two changes; and if the seventh sign be taken $-$, and the eighth $+$, there are four; but in the original equation, after the sixth term, there is only one change of sign; therefore by introducing one more positive root, there is at least one more change of sign.

Let now the alternate signs be changed by which the positive roots will be changed into negative, and conversely, the signs of the transformed equation will then be as follows:—

$$+ + - - - - + - -$$

In this series of signs there are as many changes as there were permanencies in the former, and as many permanencies as there were changes; if now the changed equation be multiplied by $x - a$, or another positive root introduced, the signs will be as follows:—

$$\begin{array}{cccccccccc}
+ & + & - & - & - & - & + & - & - & \\
 & - & - & + & + & + & + & - & + & + \\
\hline
+ & \pm & - & \mp & \mp & \mp & + & - & \mp & +
\end{array}$$

Examining the signs of this result in the same manner as in the former, it is found that the changes cannot be less than four or more than six, whereas in the transformed equation there were but three; therefore by introducing a positive root into this equation, or a negative root into the original one, there is at least one more change of sign in the transformed, or one more permanency in the original equation; hence, since the sum of the changes of sign and permanencies of sign are together equal to the number of roots, for each is equal to n, the number denoting the degree of the equation, a positive root cannot introduce more than one change, nor a negative root more than one permanency, and it has been shewn that each introduces at least one.

562. Cor. It follows, from what has been said, that every equation has as many roots as its unknown quantity has dimensions. Thus a quadratic has two roots, which are either both positive, both negative; or one positive, and one negative. A cubic has three roots, which are either all positive, all negative; two positive, and one negative; or one positive, and two negative; and the like of higher equations.

IMAGINARY ROOTS.

563. An equation, having no surds in its coefficients, may, however, have imaginary roots, which always enter in pairs of the form $a + b\sqrt{-1}$, and $a - b\sqrt{-1}$, and are called conjugate to each other; these being the roots of the quadratic equation $x^2 - 2ax + a^2 + b^2 = 0$, which contains no surds.

Since these roots can only enter such an equation in pairs, a complete equation, which has an odd number of changes of sign, has at least one real positive root; and, so far as this test is concerned, may have any odd number not greater than the number of changes of sign; but if the number of changes of sign be even, it has either two, four, or six, &c. real positive roots, or none at all. Again, if there be an odd number of permanencies of sign, the equation has at least one real negative root; but if it has an even number of permanencies, it has either two, four, or six, &c. real negative roots, or none at all.

DE GUA'S CRITERION OF IMAGINARY ROOTS.

564. If a term of an equation be wanting, and the signs of the preceding and succeeding terms be the same, it will have at least two imaginary roots.

For if the sign of the absent term be supposed different from the sign of the adjacent terms, these signs will give two changes indicating two positive roots; but if the sign of the absent term be supposed the same as that of the adjacent term, there will be two more permanencies indicating two negative roots; hence, as the roots cannot be both negative and positive, they must be imaginary.

TRANSFORMATION OF EQUATIONS.

565. To transform an equation into another which shall want the second term.

Rule. Divide the coefficient of the second term by the exponent of the highest power of x in the equation, and for x substitute y, with the quotient annexed, having an opposite sign from that of the second term of the given equation; the terms of this equation being collected, will be the equation sought, and its roots will be greater or less than the roots of the original equation by the quotient subtracted or added; for if $x = y - r$, then $y = x + r$; and if $x = y + r$, then $y = x - r$.

566. To transform an equation into another whose roots shall be either multiples or parts of the roots of the given equations.

Rule. For x substitute my or $\frac{y}{m}$, according as the roots required are to be parts or multiples of the original roots, and it will at

once appear that if $\frac{1}{m}$ represent the part that the new root is to be of the former, it is only necessary to substitute y for x, and multiply the coefficient by m raised to a power denoted by its exponent in each term; and that if the new root is to be a multiple of the original by m, it is only necessary to substitute y instead of x, and multiply the second coefficient by m, the third by m^2, and so on to the absolute term, which must be multiplied by m^n, where n denotes the degree of the equation.

For example, let the given equation be $x^4 + 6x^3 - 7x^2 + 3x - 20 = 0$; and first let it be required to find an equation whose roots are an mth part of those of the given equation, so that $x = my$, which, being substituted for x in the given equation, gives

$$m^4y^4 + 6m^3y^3 - 7m^2y^2 + 3my - 20 = 0.$$

In this equation $y = \frac{x}{m}$, and therefore its roots are an mth part of x, and it is also formed by the rule.

Next, let $y = mx$, $\therefore x = \frac{y}{m}$; and substituting this value it becomes

$$\frac{y^4}{m^4} + 6\frac{y^3}{m^3} - 7\frac{y^2}{m^2} + 3\frac{y}{m} - 20 = 0,$$

which, being multiplied by m^4, to clear it from fractions, gives

$$y^4 + 6my^3 - 7m^2y^2 + 3m^3y - 20m^4 = 0,$$

an equation which may also be formed by the rule.

567. To form an equation whose roots shall be the reciprocals of the roots of the given equation; that is, let $x = \frac{1}{y}$, and consequently $y = \frac{1}{x}$, so that for every value of x there will be a corresponding reciprocal value of y.

Taking, again, the above equation, and substituting $\frac{1}{y}$ for x, it becomes

$$\frac{1}{y^4} + \frac{6}{y^3} - \frac{7}{y^2} + \frac{3}{y} - 20 = 0.$$

Multiplying by y^4, changing the signs, and inverting the terms—that is, arranging them according to the powers of y—it becomes

$$20y^4 - 3y^3 + 7y^2 - 6y - 1 = 0;$$

an equation whose roots are evidently the reciprocals of those of

the given equation, and whose coefficients are those of the original equation in an inverted order; hence

568. RULE. Take the coefficients of the original equation, including its absolute term, and write them for the coefficient of y in an inverted order, changing all the signs if necessary, and the roots of the transformed equation will be the reciprocals of those of the original equation.

EXERCISES.

1. Change the equation $x^3 + 6x^2 + 9x - 12 = 0$, into another wanting the second term, . . . $= y^3 - 3x - 14 = 0$.

2. Change the equation $x^4 - 12x^3 + 15x^2 + 196x - 480 = 0$, into another wanting the second term, $= y^4 - 39y^2 + 70y = 0$.

3. Given the equation $x^4 - 12x^3 + 15x^2 + 196x - 480 = 0$; write the equations whose roots are double, and one-half of the roots of the given equation respectively,

$$= \begin{cases} y^4 - 24y^3 + 60y^2 + 1568y - 7680 = 0. \\ 16y^4 - 96y^3 + 60y^2 + 392y - 480 = 0. \end{cases}$$

4. Change the equation $x^3 - x^2 - 34x - 56 = 0$, into another whose roots shall be three times as great as the roots of the given equation, ... $= y^3 - 3y^2 - 306y - 1512 = 0$.

5. Change the equation $x^3 - x^2 - 34x - 56 = 0$, into another whose roots shall be the reciprocals of the given equation,

$$= 56y^3 + 34y^2 + y - 1 = 0.$$

6. What is the equation whose roots are the reciprocals of the roots of the equation $x^4 + x^3 - 16x^2 - 4x + 48 = 0$,

$$= 48y^4 - 4y^3 - 16y^2 + y + 1 = 0.$$

569. To transform an equation into another, whose roots shall be less than those of the proposed equation, by some given quantity.

RULE. Connect the given quantity by the sign $+$ with any letter different from that denoting the unknown quantity in the given equation, and it will form a sum; as $y + e = x$.

Substitute this sum and its powers for the unknown quantity, and its powers in the proposed equation, and the result will be a new equation, having its roots less by e than those of the given equation.

Let the equation, whose roots are to be reduced by e, be the following:—

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

And let $x = y + e$; then if $y + e$ be substituted for x in all the

terms of the equation, it is evident that y in the resulting equation will be less than x in the original equation by e, for $x = y + e$;

$\therefore y = x - e$. The substitution may be arranged as under—

$$\begin{aligned}
x^4 &= (y+e)^4 = y^4 + 4ey^3 + 6e^2y^2 + 4e^3y + e^4,\\
ax^3 &= a(y+e)^3 = ay^3 + 3aey^2 + 3ae^2y + ae^3,\\
bx^2 &= b(y+e)^2 = by^2 + 2bey + be^2,\\
cx &= c(y+e) = cy + ce,\\
d &= d = d;
\end{aligned}$$

$$\therefore x^4 + ax^3 + bx^2 + cx + d = y^4 + (a + 4e)y^3 + (b + 3ae + 6e^2)y^2 + (c + 2be + 3ae^2 + 4e^3)y + d + ce + be^2 + ae^3 + e^4;$$

but the first side of this equation is equal to zero, therefore

$$y^4 + (a + 4e)y^3 + (b + 3ae + 6e^2)y^2 + (c + 2be + 3ae^2 + 4e^3)y + d + ce + be^2 + ae^3 + e^4 = 0.$$

The coefficients of the resulting equation may be derived from those of the given equation, by the following very simple process:—

570. Rule. Multiply the coefficient of the first term by e, and add the product to the coefficient of the second term; again multiply this sum by e, and add the product to the coefficient of the third term; and so on to the absolute term, which will give the absolute term of the new equation; repeat the same operation on the various sums as was performed on the original coefficients, stopping at the last term but one, which will give the coefficient of y; again repeat the same process on the new sums to the last but two, and the sum will be the coefficient of y^2; repeat this process continually till the last product fall under the second coefficient, and the first coefficient, with the last sum in each of the succeeding columns taken in order, will be the coefficients and absolute term of the derived equation.

The process should be arranged as follows:—

$$\begin{array}{llll}
1+a & +b & +c & +d\,(e \text{ the multiplier.}\\
\;\;+e & +ae+e^2 & +be+ae^2+e^3 & +ce+be^2+ae^3+e^4\\
\hline
a+e & b+ae+e^2 & c+be+ae^2+e^3 & d+ce+be^2+ae^3+e^4\\
\;\;+e & \;\;ae+2e^2 & +be+2ae^2+3e^3 & \\
\cline{1-3}
a+2e & b+2ae+3e^2 & c+2be+3ae^2+4e^3 & \\
\;\;+e & +ae+3e^2 & & \\
\cline{1-2}
a+3e & b+3ae+6e^2 & & \\
\;\;+e & & & \\
\cline{1-1}
a+4e & & &
\end{array}$$

$$\therefore 1 + (a + 4e) + (b + 3ae + 6e^2) + (c + 2be + 3ae^2 + 4e^3) + (d + ce + be^2 + ae^3 + e^4)$$

are the coefficients and absolute term of the transformed equation, which has its roots less than those of the given equation by e, and they are the same as those formerly found.

This is a very important process, as it contains the substance of Horner's solution of equations of the higher degrees.

571. Cor. It is evident that an equation may be found in the same manner, whose roots shall be greater than those of the given equation by e, if the sign of e be changed from $+$ into $-$; that is, by changing in the above series the signs of all the terms which contain odd powers of e.

EXAMPLES.

1. By the above, find the equation whose roots are less by 1 than those of the equation $x^3 + 7x^2 - 5x - 12 = 0$.

$$\begin{array}{rrrl}
1 + 7 & -5 & -12 & (1 = \text{the multiplier.} \\
+1 & +8 & +3 & \\
\cline{1-3}
+8 & +3 & -9 & \\
+1 & +9 & & \\
\cline{1-2}
+9 & +12 & & \\
+1 & & & \\
\cline{1-1}
+10 & & &
\end{array}$$

$\therefore y^3 + 10y^2 + 12y - 9 = 0$ is the equation sought.

2. Find the equation whose roots are greater than those of the equation $x^4 - 3x^3 - 69x^2 + 127x + 840 = 0$ by 2.

Here, applying the Cor. (571), the multiplier 2 will be $-$, and the work as follows:—

$$\begin{array}{rrrrl}
1 - 3 & -69 & +127 & +840 & (-2 = \text{the multiplier.} \\
-2 & +10 & +118 & -490 & \\
\cline{1-4}
-5 & -59 & +245 & +350 & \\
-2 & +14 & +90 & & \\
\cline{1-3}
-7 & -45 & +335 & & \\
-2 & +18 & & & \\
\cline{1-2}
-9 & -27 & & & \\
-2 & & & & \\
\cline{1-1}
-11 & & & &
\end{array}$$

$\therefore y^4 - 11y^3 - 27y^2 + 335y + 350 = 0$ the required equation.

3. Diminish the roots of $x^4 - 10x^3 + 4x^2 - 20x - 78 = 0$, by a quantity greater than the greatest positive root.

By the above process the roots may be diminished by 1, and those of the resulting equation again by 1, or any greater number,

till the signs of the coefficients and the absolute term be all positive, then (560) the roots will all be negative; and hence the roots will have been diminished by a quantity greater than the greatest positive one.

572. If, in the course of the operation, the absolute term should become zero, then the quantity by which the roots have then been diminished is a root of the equation. The operation will be as follows:—

$$
\begin{array}{rrrr}
1-10 & +4 & -20 & -78(1 \\
+1 & -9 & -5 & -25 \\
\hline
-9 & -5 & -25 & -103 \\
+1 & -8 & -13 & \\
\cline{1-3}
-8 & -13 & -38 & \\
+1 & -7 & & \\
\cline{1-2}
-7 & -20 & & \\
+1 & & & \\
\cline{1-1}
-6 & & & \\
\hline
1-6 & -20 & -38 & -103(8 \\
+8 & +16 & -32 & -560 \\
\hline
+2 & -4 & -70 & -663 \\
+8 & +80 & +608 & \\
\cline{1-3}
+10 & +76 & +538 & \\
+8 & +144 & & \\
\cline{1-2}
+18 & +220 & & \\
+8 & & & \\
\cline{1-1}
+26 & & & \\
\hline
1+26 & +220 & +538 & -663(1 \\
+1 & +27 & +247 & +785 \\
\hline
+27 & +247 & +785 & +122 \\
+1 & +28 & +275 & \\
\cline{1-3}
+28 & +275 & +1060 & \\
+1 & +29 & & \\
\cline{1-2}
+29 & +304 & & \\
+1 & & & \\
\cline{1-1}
+30 & & & \\
1+30 & +304 & +1060 & +122;
\end{array}
$$

$\therefore\ y^4 + 30y^3 + 304y^2 + 1060y + 122 = 0$ is the equation sought.

And since the roots have been diminished by $1 + 8 + 1 = 10$, the greatest positive root of the equation is less than 10; but when the roots were diminished by 9, there remained one change of sign indicating one real positive root; therefore the above equation has at least one positive root, which lies between 9 and 10.

EXERCISES.

1. Find the equation whose roots are less by 1 than those of the equation $x^3 + 7x^2 - 4x - 12 = 0$, $= y^3 + 10y^2 + 13y - 8 = 0$.

2. Find the equation whose roots are less by 2 than those of the equation $x^4 - 10x^3 + 4x^2 - 5x + 24 = 0$,
$$= y^4 - 2y^3 - 32y^2 - 77y - 34 = 0.$$

3. Find the equation whose roots are greater than the roots of the equation $x^4 - 4x^3 - 19x^2 + 46x + 120 = 0$, by 4,
$$= y^4 - 20y^3 + 125y^2 - 250y + 144 = 0.$$

4. Find the equation whose roots are less by 3 than those of the equation $x^5 - 12x^4 + 7x^3 - 30x^2 + 51x - 40 = 0$,
$$= y^5 + 3y^4 - 47y^3 - 345y^2 - 831y - 697 = 0.$$

5. Find a number greater than the greatest positive root of the equation $x^3 - 9x^2 + 26x - 23 = 0$, ... $= 4$.

6. Form the equation whose roots are greater by 2 than those of the equation $2x^4 - 5x^2 + 3x - 1 = 0$,
$$= 2y^4 - 16y^3 + 43y^2 - 41y + 5 = 0.$$

BUDAN'S THEOREM FOR THE DISCOVERY OF IMAGINARY ROOTS.

573. If the roots of an equation be reduced by any quantity r, and in performing the operation there be m changes of sign lost; and if, in reducing the roots of the reciprocal equation by $\frac{1}{r}$, there be n changes of sign left, then between the interval r and 0 there are at least $m - n$ imaginary roots.

For in reducing the roots of the equation by r, all positive roots less than r will have become negative, and there will be as many positive roots between 0 and r as there are positive roots changed into negative—that is, as there are changes of sign lost; whereas in reducing the reciprocal equation by $\frac{1}{r}$, which must be less than any of its positive roots between $\frac{1}{r}$ and ∞, no changes should be lost; but if there are imaginary roots, changes may be lost; and the changes of sign left in this equation, deducted from the changes lost in the former operation, will indicate the number of imaginary roots between the assumed limits; that is, $m - n =$ the imaginary roots.

574. Cor. If the reducing quantity r be greater than the greatest positive root, all the roots of the depressed equation will be negative, and the changes of sign will all be lost; whereas in reducing the reciprocal equation by $\frac{1}{r}$, all the changes ought to remain if the positive roots are real; then those changes that still remain, deducted from those originally lost, will leave the number of imaginary roots that are apparently positive. If the signs of the alternate terms be changed, the negative roots will be changed into positive, and the positive into negative; and if the same operation be performed on the equation so changed, the number of apparently negative imaginary roots will be discovered.*

EXAMPLE.

Find, by Budan's and De Gua's theorems, the number of imaginary roots of the equation $x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0$.

Since this equation is of an odd degree, and its absolute term is negative, it has at least one real positive root, and as there is only one change of sign, it cannot have more than one; therefore, changing the signs of the alternate terms, we apply Budan's theorem to ascertain the number of imaginary negative roots. The equation thus changed becomes $x^5 - 3x^4 + 2x^3 + 3x^2 - 2x + 2 = 0$.

Coefficients of the Given Equation so changed.

$$
\begin{array}{rrrrrrl}
1 & -3 & +2 & +3 & -2 & +2 & (1 \\
 & +1 & -2 & \pm 0 & +3 & +1 & \\
\hline
 & -2 & \pm 0 & +3 & +1 & +3 & \\
 & +1 & -1 & -1 & +2 & & \\
\hline
 & -1 & -1 & +2 & +3 & & \\
 & +1 & \pm 0 & -1 & & & \\
\hline
 & \pm 0 & -1 & +1 & & & \\
 & +1 & +1 & & & & \\
\hline
 & +1 & \pm 0 & & & & \\
 & +1 & & & & & \\
\hline
 & +2 & & & & & \\
\hline
1 & +2 & \pm 0 & +1 & +3 & +3 &
\end{array}
$$

Coefficients of the Reciprocal Equation.

$$
\begin{array}{rrrrrrl}
2 & -2 & +3 & +2 & -3 & +1 & (1 \\
 & +2 & \pm 0 & +3 & +5 & +2 & \\
\hline
 & \pm 0 & +3 & +5 & +2 & +3 & \\
 & +2 & +2 & +5 & +10 & & \\
\hline
 & +2 & +5 & +10 & +12 & & \\
 & +2 & +4 & +9 & & & \\
\hline
 & +4 & +9 & +19 & & & \\
 & +2 & +6 & & & & \\
\hline
 & +6 & +15 & & & & \\
 & +2 & & & & & \\
\hline
 & +8 & & & & & \\
\hline
2 & +8 & +15 & +19 & +12 & +3 &
\end{array}
$$

* On the preceding theorem, together with that in (564), which is De Gua's criterion, the late Professor Davies remarks, in the appendix to the *Ladies' Diary* of 1839: 'I have found the criteria of De Gua and Budan quite equal to the detection of imaginary roots of every algebraical equation to which I have applied them.'

Here, in depressing the roots of the direct equation by 1, there are 4 variations of sign lost; for, by De Gua's criterion, since the third coefficient is zero, and the adjacent terms have the same sign, this indicates two imaginary roots; hence that term may be taken with a $+$ sign, and therefore there are four variations lost; and in the reciprocal equation, when depressed, there are no variations left; $\therefore 4 - 0 = 4$ is the number of imaginary negative roots.

EXERCISES.

1. Find how many roots are real, and how many are imaginary in the equation $x^5 - 10x^3 + 6x + 1 = 0$,

The roots are all real; 2 positive, and 3 negative.

2. Has the equation $x^4 - 4x^3 + 8x^2 - 16x + 20 = 0$ any real roots? . . . . None.

3. How many roots of the equation $x^5 - x^4 - 15x^3 + 38x^2 - 26x + 6 = 0$ are real, and how many imaginary? All are real.

4. How many real and imaginary roots has the equation $x^4 + 12x^3 + 29x^2 - 16x + 2 = 0$, ... All are real.

DEVELOPMENT OF AN INTEGRAL FUNCTION OF A BINOMIAL $x + y$.

575. Let the integral function of x be

$$fx = Ax^n + Bx^{n-1} + Cx^{n-2} + ; \ldots$$

by writing $x + y$ instead of x, it becomes

$$f(x + y) = A(x + y)^n + B(x + y)^{n-1} + C(x + y)^{n-2} + ; \ldots$$

and if we develop the powers of the binomial $x + y$, according to the decreasing powers of x, we find

$$\begin{array}{l|l|l} A(x+y)^n = Ax^n + nAx^{n-1} & y + n(n-1)Ax^{n-2} & \dfrac{y^2}{1.2} +, \\ +B(x+y)^{n-1} = +Bx^{n-1} + (n-1)Bx^{n-2} & +(n-1)(n-2)Bx^{n-3} & \\ +C(x+y)^{n-2} = +Cx^{n-2} + (n-2)Cx^{n-3} & +(n-2)(n-3)Cx^{n-4} & \&c. \\ + & & \end{array}$$

Let us now put for abridgment

$$X = Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots$$

$$X' = nAx^{n-1} + (n - 1)Bx^{n-2} + (n - 2)Cx^{n-3} + \ldots$$

$$X'' = n(n - 1)Ax^{n-2} + (n - 1)(n - 2)Bx^{n-3} + (n - 2)(n - 3)Cx^{n-4} + \ldots$$

Then the preceding result will be expressed as follows :—

$$f(x+y) = X + X'y + \frac{X''}{1\,.\,2}y^2 + \frac{X'''}{1\,.\,2\,.\,3}y^3 + \dots$$

Now X is the original expression, which was represented by fx; X' is derived from X by multiplying each of the terms by the exponent of x in that term, and diminishing the exponent by unity; X'' is derived in the same way from X'—namely, by multiplying each term of X' by the exponent of x in the term, and diminishing the exponent by unity; in the same manner, by extending the expansion to more terms, it can be shewn that X''' is derived from X'', in the same manner as X' from X.

576. The polynomial X' is called the *derivative function* of X; and X'' is called the *derivative function* of X', or the *second derivative function* of X; and so on. X' is also called the limiting ratio of the function X, and is a quantity of very considerable importance in the theory of equations, being the same as the first differential coefficient, which becomes the object of research in the differential calculus, and also identical with the coefficient of y, as found by Rule (570), if x be put for e. Since X has been designated by $f(x)$, X', X'', X''', will be very appropriately designated by $f'(x)$, $f''(x)$, $f'''(x)$; and using this notation, we obtain for the development sought,

$$f(x+y) = f(x) + f'(x)y + f''(x)\frac{y^2}{1.2} + f'''(x)\frac{y^3}{1\,.\,2\,.\,3} + \dots$$

As the exponent of x diminishes by unity in passing from the given polynomial to its derived function, or from any derived function to the following, a polynomial of the nth degree will have n derivative functions, of which the last will not contain x. It is easy to see, besides, that if the first term of the polynomial is represented by Ax^n, as above, the nth derivative function will be

$$1\,.\,2\,.\,3\,.\,4\,\dots\,n\,.\,A.$$

Consequently, the law of the terms which compose the development of $f(x+y)$ gives for the last term Ay^n, which may be seen otherwise; for it is clear that, after the substitution of $x+y$ instead of x in the polynomial $Ax^n + Bx^{n-1} + Cx^{n-2} +, \dots$ the quantity Ax^n, which becomes $A(x+y)^n$, gives the term Ay^n, and that there is no other term in which the exponent of y can be either equal to or greater than n.

577. To apply to a practical example what has been shewn concerning the derived functions, let

$$f(x) = x^5 + 5x^4 + x^3 - 16x^2 - 20x - 16.$$

If we wish to find what the above polynomial becomes when x is replaced by $y-1$, we will calculate the successive derivatives.

The first derivative is

$$f'(x) = 5x^4 + 20x^3 + 3x^2 - 32x - 20.$$

In forming the second derivative, and dividing it by 2, we have

$$\frac{f''(x)}{1.2} = 10x^3 + 30x^2 + 3x - 16.$$

The derivative of this last function, divided by 3, is

$$\frac{f'''(x)}{1.2.3} = 10x^2 + 20x + 1.$$

The derivative of this, again, divided by 4, is

$$\frac{f^{\text{IV}}(x)}{1.2.3.4} = 5x + 5.$$

Repeating the same process, and dividing by 5, we have

$$\frac{f^{\text{V}}(x)}{1.2.3.4.5} = 1.$$

By substituting in these various functions -1 for x, we find

$f(-1) = -9$, $f'(-1) = 0$, $\dfrac{f''(-1)}{1.2} = 1$, $\dfrac{f'''(-1)}{1.2.3} = -9$, $\dfrac{f^{\text{IV}}(-1)}{1.2.3.4} = 0$, $\dfrac{f^{\text{V}}(-1)}{1.2.3.4.5} = 1$; and consequently

$$f(y-1) = y^5 - 9y^3 + y^2 - 9.$$

Employing the same process to find what the polynomial $2x^4 - 5x^2 + 3x - 1$ becomes, when $y - 2$ is substituted for x, we obtain

$$2y^4 - 16y^3 + 43y^2 - 41y + 5.$$

VALUES OF AN INTEGRAL FUNCTION OF x FOR VERY GREAT OR VERY SMALL VALUES OF x. CONTINUITY OF INTEGRAL FUNCTIONS OF ONE VARIABLE.

578. If in the polynomial $Ax^m + Bx^n + Cx^p +$, &c. the exponents m, n, p, &c. being positive integral numbers forming a decreasing series, we give to x numerical values sufficiently great, either positive or negative, the values of the polynomial will have the same sign as that of its first term Ax^m; and a value can be given to x so great, that the value of the polynomial may be as great as we please.

The polynomial given above may be written as follows:—

$$Ax^m\left(1 + \frac{B}{A}\cdot\frac{1}{x^{m-n}} + \frac{C}{A}\cdot\frac{1}{x^{m-p}} +, \text{\&c.}\right)$$

For a very great value of x, the fraction $\frac{1}{x^{m-n}}$, $\frac{1}{x^{m-p}}$, &c. will all have values very small, consequently the sum of the terms $\frac{B}{A} \cdot \frac{1}{x^{m-n}}$, $\frac{C}{A} \cdot \frac{1}{x^{m-p}}$, &c. of which the number is necessarily limited, will also have a value very small; so that, if x is made sufficiently great, the polynomial enclosed in parentheses will have a value differing very little from unity, and which will consequently be positive. The given polynomial will therefore take values of the same sign as the term Ax^m. We see also that the value of the polynomial may be made as great as we wish, since the factor Ax^m increases indefinitely with x.

579. If in the polynomial $Ax^m + Bx^n + Cx^p +$, &c. the exponents m, n, p, &c. being positive integral numbers, which form an increasing series, and the polynomial being composed of a limited number of terms, we give to x a very small value either positive or negative, the polynomial will have a very small value, of the same sign as that of its first term Ax^m.

The given polynomial may be written as follows:—

$$Ax^m(1 + \frac{B}{A}x^{n-m} + \frac{C}{A}x^{p-m} + \&c.)$$

For a very small value of x, the quantities x^{n-m}, x^{p-m}, &c. of which the exponents are integral and positive, will all have values very small; so that if we make x converge towards zero, the polynomial enclosed in parentheses will take a value differing very little from unity, and which will consequently be positive. The given polynomial will then take values of the same sign as that of its first term Ax^m. We see also that the value of the polynomial can be made as small as we please, since the factor Ax^m decreases indefinitely with x.

580. Having given an integral function $f(x)$, and a particular value a of x, a quantity h can be found so small, that the difference $f(a+h) - f(a)$ shall be less than any given quantity, however small that may be.

For, if in (576) x and y be replaced by a and h, and $f(a)$ transposed to the first side, we have

$$f(a+h) - f(a) = f'(a)h + f''(a)\frac{h^2}{1.2} + f'''(a)\frac{h^3}{1.2.3} + \cdots$$

Now, if a very small value be given to h, each of the terms $f'(a)h$, $f''(a)\frac{h^2}{1.2}$, &c. will have a very small value; and as the number of these terms is limited, h can be conceived so small, that

the sum of all these terms may be as small as we please, which establishes the proposition.

581. Scholium. Let α and β be two given numbers, β being greater than α. If we wish to determine the quantity h in such a manner that for any value a of x, comprised between α and β, we may have $f(a+h)-f(a) \angle \varepsilon$, indicating by ε a quantity as small as we please, it may be obtained as follows:—

Putting C for a number greater than any of the coefficients which enter into the polynomials $f'(x), \frac{f''(x)}{1.2}, \frac{f'''(x)}{1.2.3}$, &c.; each of these coefficients, for a value of x comprised between α and β, will be less than $C\beta^{m-1}$, if $\beta > 1$, and less than C, if $\beta \angle 1$; consequently the value of each polynomial will be less than $mC\beta^{m-1}$, less than mC, according as $\beta > 1$, or $\beta \angle 1$.

This being established, writing K for the number $mC\beta^{m-1}$, if $\beta > 1$, and for the number mC, if $\beta \angle 1$, and developing the function $f(a+h)-f(a)$, we have, for any value a of x, comprised between α and β,

$$f(a+h)-f(a) \angle Kh(1+h+h^2+\dots h^{m-1}).$$

The second member of this inequality is equal to $\frac{Kh(1-h^m)}{1-h}$; consequently, if we suppose $h \angle 1$, this second member will be less than $\frac{Kh}{1-h}$. If now $\frac{Kh}{1-h} \angle \varepsilon$, still more will $f(a+h)-f(a) \angle \varepsilon$.

Now the first condition is satisfied when $h \angle \frac{\varepsilon}{K+\varepsilon}$. If then values be given to x, increasing from α up to β, and such that the difference of two consecutive values may be equal to or less than the quantity $\frac{\varepsilon}{K+\varepsilon}$, we will obtain values of $f(x)$ such that the difference of two consecutive values will be less than ε.

582. When two numbers α and β, being substituted for x in the first member of an equation $f(x)=0$, give results with contrary signs, the equation has at least one real root comprised between α and β.

From what has been established in the preceding article, we can give to x values, increasing from α up to β, such that the difference between each of the corresponding values of $f(x)$ and the following may be less than a quantity ε as small as we please. There are necessarily two consecutive values of $f(x)$ which have contrary signs, since, by hypothesis, the two extreme values $f(\alpha)$ and $f(\beta)$ have contrary signs. These two consecutive values of $f(x)$, which

have contrary signs, and differ from each other by a quantity less than ε, are each numerically less than ε. Now this conclusion holds true, however small ε may be; and hence there exists between α and β at least one value of x, for which $f(x)$ is zero, or $f(x) = 0$.

SCHOLIUM. As it is possible that the polynomial $f(x)$ may pass several times from positive to negative, or from negative to positive, while x varies from α to β, the equation $f(x) = 0$ may have several real roots comprised between α and β.

583. An equation of an *odd* degree has at least one real root, of which the sign is contrary to that of its last term.

Let $x^m + Ax^{m-1} + Bx^{m-2} \dots \pm K = 0$, in which m, the exponent of the highest power of x, is an odd number.

When the sign of the absolute term is negative, if in the first member of the equation x be made equal to zero, the sign of the result will be negative; for it will be no other than the last term. Again, if a positive value be given to x so great that the sign of the first member will be the same as that of its first term, the sign of the result will be positive.

The equation has then at least one positive root.

If the last term be positive, and x be made equal to zero, the result will be positive; but if a negative value be given to x sufficiently great (578), the result will be negative, and the equation has in this case at least one negative root.

584. An equation of an *even* degree, of which the last term is negative, has at least *two real roots*—the one positive, and the other negative.

For by making $x = 0$, the result will be negative; and if a value be given to x sufficiently great, whether that value be positive or negative, the result will be positive, since it will have the sign of the first term (578), which, being of an even degree, will always be positive.

585. If a is a root of the equation $f(x) = 0$, the first member of the equation is divisible by $x - a$.

Let the general equation $f(x) = 0$ be of the form

$$x^m + A_1x^{m-1} + A_2x^{m-2} + \dots + A_{m-1}x + A_m = 0.$$

If a is a root of this equation, we have the equality,

$$a^m + A_1a^{m-1} + A_2a^{m-2} + \dots + A_{m-1}a + A_m = 0.$$

From this latter equation, it is evident that

$$A_m = - a^m - A_1a^{m-1} - A_2a^{m-2} - \dots - A_{m-1}a.$$

Substituting this value of A_m in the original equation, and

arranging the powers of a along with the corresponding powers of x, we obtain the following :—

$$x^m - a^m + A_1(x^{m-1} - a^{m-1}) + A_2(x^{m-2} - a^{m-2}) \dots + A_{m-1}(x - a) = 0.$$

But $x - a$ is a divisor of each of the binomials $(x^m - a^m)$, $x^{m-1} - a^{m-1}$, &c. (87, THEO. II.) The first member of the equation $f(x) = 0$ is therefore divisible by $x - a$, when a is a root of that equation.

To obtain the quotient, it is only necessary to divide each of the binomials $x^m - a^m$, $x^{m-1} - a^{m-1}$, &c. by $x - a$, and to add afterwards the partial quotients, multiplying the second by A_1, the third by A_2, the fourth by A_3, &c. and the result becomes the following :—

$$\begin{array}{l|l|l|l}
x^{m-1} + \;\; a & x^{m-2} + \;\; a^2 & x^{m-3} + \;\; a^3 & x^{m-4} \dots + \;\; a^{m-1} \\
\quad\;\; + A_1 & \quad\;\; + A_1 a & \quad\;\; + A_1 a^2 & \quad\;\; \dots + A_1 a^{m-2} \\
 & \quad\;\; + A_2 & \quad\;\; + A_2 a & \quad\;\; \dots + A_2 a^{m-3} \\
 & & \quad\;\; + A_3 & \quad\;\; \dots + A_3 a^{m-4} \\
 & & & \quad\;\; \dots \\
 & & & \quad\;\; \dots \\
 & & & \quad\;\; \quad\;\; + A_{m-1}
\end{array}$$

The coefficients of the terms of the quotient can be obtained, beginning at the second, by multiplying the coefficient of the preceding term by a, and adding to the product the coefficient of that term in the original equation, which is of the same rank as the term of the quotient which is to be formed. The coefficient of the first term of the quotient is the same as that of the first term of the original polynomial.

LIMITS OF THE ROOTS OF AN EQUATION.

586. If a number be determined such that by putting it, or any greater number, instead of x in the first side of an equation, the result will be positive; it is clear that it will be a superior limit of the roots.

Let the equation

$$x^m \pm Ax^{m-1} \pm Bx^{m-2} \pm Cx^{m-3} \dots \pm K = 0$$

have the first term x^m positive, and the other terms indifferently positive or negative; if N indicate the greatest negative coefficient, the first side will be rendered positive by satisfying the following inequality :—

$$x^m > Nx^{m-1} + Nx^{m-2} + \dots + Nx + N;$$

the second side $= N(x^{m-1} + x^{m-2} + \dots + x + 1) = \dfrac{N(x^m - 1)}{x - 1}$.

The above condition is then

$$x^m > \frac{N(x^m - 1)}{x - 1},$$

which will be verified if $x - 1 = N$, or $x = 1 + N$, and still more if $x > 1 + N$; hence in every equation a superior limit of the roots may be obtained by adding unity to the greatest negative coefficient of the first member of the equation. It is also evident, that since by changing the signs of the alternate terms of an equation, the positive roots become negative, and the negative roots positive; if a negative value be given to x either equal to or greater than one +, the greatest negative coefficient of the equation so changed, an inferior limit to the roots will be obtained.

587. When the term of degree immediately inferior to that of the equation is not negative, a limit less than the preceding may be obtained.

Let the equation be

$$x^m \ldots - Fx^{m-n} \pm Gx^{m-n-1} \ldots \pm K = 0,$$

the term $- Fx^{m-n}$ being the first negative term, and the following terms being indifferently positive or negative.

Designating by N the absolute value of the greatest negative coefficient; the first member of the equation will be rendered positive if the following inequality be satisfied:—

$$x^m > Nx^{m-n} + Nx^{m-n-1} \ldots + N, \text{ or } x^m > \frac{Nx^{m-n+1} - 1}{x - 1}.$$

If we suppose $x > 1$, it will be sufficient that we have

$$x^m > \frac{Nx^{m-n+1}}{x - 1}, \text{ or } x^{n-1}(x - 1) > N.$$

Also the last condition will evidently be fulfilled if we have

$$(x - 1)^{n-1}(x - 1) = \text{or} > N; \text{ that is, } (x - 1)^n = \text{or} > N;$$

whence $$x = \text{or} > 1 + \sqrt[n]{N}.$$

A superior limit to the roots can therefore be obtained by adding unity to the root of the absolute value of the greatest negative coefficient, of which the exponent is the difference between the degree of the equation and the exponent of the first negative term. If, however, the coefficient N be less than unity, the limit $1 + N$ formerly obtained will be preferable to that which has been obtained in this article.

THEORY OF EQUAL ROOTS.

588. When an equation has equal roots, it may be reduced to several other equations more simple, in which each root is only found once. To discover these equations will be the object of the present article.

Let the equation be

$$x^m + Ax^{m-1} + Bx^{m-2} + \dots + Hx + K = 0.$$

Representing by $f(x)$ the first member of this equation, and by $f'(x)$ its derivative function, or the polynomial.

$$mx^{m-1} + (m-1)Ax^{m-2} + (m-2)Bx^{m-3} + \dots + H.$$

The polynomial $f'(x)$ is the coefficient of the first power of y in the result which was obtained by writing $x + y$ in place of x in the polynomial $f(x)$ (575); now, in representing the roots of the equation by $a, b, c, \dots k$, we have

$$f(x) = (x-a)(x-b)(x-c)\dots(x-k);$$

then, by writing $x + y$ in place of x,

$$f(x+y) = (y+x-a)(y+x-b)(y+x-c)\dots(y+x-k).$$

The second member of the last equality may be regarded as the product of m binomial factors, which have for their first term y, and of which the second terms are $x - a$, $x - b$, &c. Consequently the coefficient of the first power of y in this product is the sum of all the products of the quantities $x - a$, $x - b$, &c. taken $m - 1$ at a time. Besides, to form these products, it is sufficient to divide successively $f(x)$ by each of the factors $x - a$, $x - b$, &c. We have then

$$f'(x) = \frac{f(x)}{x-a} + \frac{f(x)}{x-b} + \frac{f(x)}{x-c} \dots + \frac{f(x)}{x-k}.$$

Let now $f(x) = (x-a)^n(x-b)^p(x-c)^q \dots$

To apply this to the proposition which is to be established, the quotient of $f(x)$ by $x - a$, must be repeated n times, for there are n factors $x - a$, the quotient of $f(x)$ by $x - b$ must be repeated p times, the quotient of $f(x)$ by $x - c$ must be repeated q times, &c.; thus we obtain

$$\begin{aligned} f'(x) = {} & n(x-a)^{n-1}(x-b)^p(x-c)^q \dots \\ & + p(x-a)^n(x-b)^{p-1}(x-c)^q \dots \\ & + q(x-a)^n(x-b)^p(x-c)^{q-1} \dots \\ & + \dots \end{aligned}$$

Putting $D = (x-a)^{n-1}(x-b)^{p-1}(x-c)^{q-1}, \dots$

D divides $f(x)$ and $f'(x)$. It is the greatest common divisor of

these two quantities, otherwise it would be necessary that one of the factors of $f(x)$ should divide the quotient of $f'(x)$ by D, which is
$$n(x-b)(x-c)\ldots+p(x-a)(x-c)\ldots$$
$$+\,q(x-a)(x-b)\ldots+\ldots$$
Now, each of the factors $x-a$, $x-b$, &c. divides all the parts of this sum, with the exception of one; consequently the sum is not divisible by any of these factors.

589. Therefore, when an equation has equal roots, the first member of the equation and its derivative function have an algebraic common divisor, and this common divisor is the product of all the factors corresponding to the equal roots of the equation, each raised to a power less by unity than its power in $f(x)$.

When the equation has no equal roots, the first member and its derivative function have not any common algebraic factor; for the exponents n, p, q, &c. being unity, the greatest common divisor of $f(x)$ and $f'(x)$ is reduced to a number.

STURM'S THEOREM.

590. Let $V=0$ be an equation of any degree of which all the roots are unequal, and let V_1 be the derivative function or limiting equation of V. We operate as if the object were to find the greatest common divisor of V and V_1, with this single difference, that we must change the signs of all the remainders before they be taken for divisors. This change of signs, which would be a matter of indifference if we only wished to find the greatest common divisor, is essential in the present case. Indicating by Q_1 the quotient of the division of V by V_1, and by $-V_2$, the corresponding remainder, we will have
$$V=V_1Q-V_2.$$
Dividing V_1 by V_2—that is to say, by the remainder of the first operation taken with a contrary sign, and indicating by Q_2 the quotient, and by $-V_3$, the corresponding remainder, we will have again
$$V_1=V_2Q_2-V_3.$$
By continuing thus, we will necessarily arrive at a numerical remainder $-V_r$, since we have supposed that the equation $V=0$ has no equal roots (588), and we will have this series of relations:
$$\begin{aligned}
V\;\; &= V_1Q_1 - V_2,\\
V_1 &= V_2Q_2 - V_3,\\
V_2 &= V_3Q_3 - V_4,\\
&\vdots\\
V_{n-1} &= V_nQ_n - V_{n+1},\\
&\vdots\\
V_{r-2} &= V_{r-1}Q_{r-1} - V_r.
\end{aligned}$$

If multipliers be introduced in the course of the operation to avoid fractional quotients, these multipliers must be positive, in order to avoid a change of sign in the remainder.

The above relations being established, the consideration of the functions V, V_1, V_2, ... has led M. Sturm to the following theorem:—

591. THEOREM. When any two numbers α and β positive or negative, of which α is less than β, are substituted instead of x in the series of functions V, V_1, V_2, ..., V_{r-1}, V_r, the number of variations of the series of signs of these functions for $x = \beta$ will be at most equal to the number of variations of the series of signs of these same functions for $x = \alpha$; and if it is less, the difference will be equal to the number of real roots of the equation $V = 0$ comprised between α and β.

To demonstrate this theorem, it is necessary to examine how the number of variations formed by the signs of the functions V, V_1, V_2, ..., V_r, disposed in the order indicated, for any value of x, can alter when x passes through different degrees of magnitude. Now, there can only be a change in this series of signs by making x increase, so that one of the functions V, V_1, V_2, &c., may change its sign, and consequently become zero.

It presents, then, two cases to examine, according as the function which becomes zero is the first V, or one of the intermediate V_1, V_2, &c.; for it is not the last function V_r which can become zero, since V_r is a number.

CASE I. To examine the change which takes place in the series of signs, when x, increasing by insensible degrees, attains and passes a value which renders V equal to zero. If that value of x, which we designate by a, be substituted in the derivative function V_1, that function will become a number positive or negative, since, by hypothesis, the equation $V = 0$ has no equal roots. Representing by u a positive quantity so small that the equation $V_1 = 0$ may not have a root comprised between $a - u$ and $a + u$; V_1 will have the same sign when $x = a - u$, $x = a$ and when $x = a + u$.

This being established, let us designate for a moment V by $f(x)$ and V_1 by $f'(x)$, we will have, by observing that $f(a) = 0$,

$$f(a - u) = -\frac{u}{1}f'(a) + \frac{u^2}{1.2}f''(a) - \frac{u^3}{1.2.3}f'''(a) + \dots$$

As there is no limit to the smallness of u, we can make u so small, that the sign of the development of $f(a - u)$ may depend on the sign of the first term (579); so that $f(a - u)$ will have the same sign as $-uf'(a)$, and consequently it will have a sign contrary to that of $f'(a)$; now, $f'(a)$ and $f'(a - u)$ have the same

sign, therefore $f(a-u)$ and $f'(a-u)$ have contrary signs. Hence V and V_1 will have contrary signs for $x=a-u$.

By changing $-u$ into $+u$ in the preceding development, we have

$$f(a+u)=\frac{u}{1}f'(a)+\frac{u^2}{1\,.\,2}f''(a)+\frac{u^3}{1\,.\,2\,.\,3}f'''(a)+\dots;$$

and we see here, the same as above, that $f(a+u)$ will have the same sign as $f'(a)$, and consequently the same sign as $f'(a+u)$. Hence V and V_1 have the same sign for $x=a+u$.

Therefore, if for $x=a$ the sign of $f'(x)$ or of V_1 is $+$, the sign of V will be $-$ for $x=a-u$, and it will be $+$ for $x=a+u$. If, on the contrary, the sign of V_1 is $-$ for $x=a$, the sign of V will be $+$ for $x=a-u$, and $-$ for $x=a+u$. These results lead to the construction of the following table:—

$$\text{For}\left\{\begin{array}{lcc} & V & V_1 \\ x=a-u & - & + \\ x=a & 0 & + \\ x=a+u & + & + \end{array}\right\}\quad\text{or}\quad\left\{\begin{array}{cc} V & V_1 \\ + & - \\ 0 & - \\ - & - \end{array}\right.$$

Consequently, when a is a root of the equation $V=0$, the sign of V forms with the sign of V_1 one *variation* before x attains the value a, and this variation is changed into a *permanency* after x has passed that value.

As to the other functions V_2, V_3, &c. each of them will have, like V_1, either for $x=a-u$, or for $x=a+u$, the same sign as they have for $x=a$, if none of them vanish for $x=a$, at the same time as V. We will now examine what will take place when one of these functions vanishes.

CASE II. Let V_n be the intermediate function which becomes zero for $x=b$. This value of x neither reduces to zero the function V_{n-1} which precedes V_n, nor the function V_{n+1} which immediately follows it; for if it did, the factor $x-b$ would divide at the same time two consecutive remainders, V_{n-1} and V_n, or V_n and V_{n+1}; consequently $x-b$ would be a multiple factor of the polynomial V, which is impossible, since we have supposed that the equation $V=0$ has no equal roots. Besides, from the equality $V_{n-1}=V_nQ_n-V_{n+1}$, which is one of the relations of (590), V_n being nothing for $x=b$, we have $V_{n-1}=-V_{n+1}$; therefore V_{n-1} and V_{n+1} have contrary signs when $x=b$.

This being established, by substituting instead of x two numbers, $b-u$ and $b+u$, very little different from b; the functions V_{n-1} and V_{n+1} will have for these two values of x the same signs as they have for $x=b$, since u can be taken so small that neither of the functions V_{n-1} nor V_{n+1} shall change its sign while x passes

the interval from $b-u$ to $b+u$. It follows from thence, that whatever may be the sign of V_n for $x=b-u$, as it is placed between the signs of V_{n-1} and V_{n+1}, which have contrary signs, the signs of the three consecutive functions V_{n-1}, V_n, and V_{n+1}, when $x=b-u$ will always form a permanency and a variation, or a variation and a permanency. It can be proved in the same manner, that whatever may be the sign of V_n for $x=b+u$, the signs of the three consecutive functions V_{n-1}, V_n, and V_{n+1}, when $x=b+u$, can only present one variation.

Thus, the series of signs of all the functions V_1, ..., V_r, for $x=b+u$, will contain precisely as many variations as the series of signs of these functions for $x=b-u$. Therefore, when any intermediate function passes through zero, the number of variations in the series of signs is not changed, unless the value of x, which reduces this intermediate function to nothing, should also reduce to zero the first function V; in this case, the change of sign will cause one variation to disappear from the left of the series of signs, as was proved in the first case.

It is clear that the same conclusion will subsist if several intermediate functions, not adjacent, become nothing for the value $x=b$.

It is therefore demonstrated, that each time that the variable x, increasing by insensible degrees, attains and passes a value which renders $V=0$, the series of signs of the functions V, V_1, V_2, ..., V_r, loses one variation formed by the signs of V and of V_1, which is replaced by a permanency; while the changes of signs of the intermediate functions V_1, V_2, ..., V_{r-1}, can never either augment or diminish the number of the variations. Consequently, if we take any number α, positive or negative, and another number β greater than α, and if we make x increase from α up to β, whatever number of values of x there are comprised between α and β which will reduce V to zero, so many will the series of signs of the functions V, V_1, V_2, ..., V_r, for $x=\beta$, contain variations less than the series of signs of these functions for $x=\alpha$. The principle which has just been stated, is no other than the theorem which was to be established, expressed in other words.

592. SCHOLIUM. It may happen that one of the functions V_1, V_2, V_3, ..., V_{r-1}, becomes nothing, either for $x=\alpha$, or for $x=\beta$. In this case, it is sufficient to consider the variations of the series of the signs of all the functions, without having regard to that which vanishes. For it has been shewn, that when the function V_n becomes nothing for $x=\alpha$, if, instead of x, a quantity very little different from α be substituted, the signs of the three functions V_{n-1}, V_n, V_{n+1}, will always present one variation and one permanency; and the variation will still be found when the function V_n is omitted.

593. Cor. I. To know the number of real roots of an equation $V=0$, it is necessary to substitute, in all the functions, two quantities, α and β, between which all the roots may be comprised. Now, it is always possible to give to x a value so great that each of the functions $V, V_1, V_2, \ldots, V_r$, may have the sign of its first term (578); therefore, if we consider at first the signs of the first terms of the functions by supposing x negative, and afterwards the signs of these same terms by supposing x positive, the excess of the number of variations of the first series of signs above the number of variations of the second series will be precisely the number of real roots of the equation.

594. Cor. II. The theorem of M. Sturm furnishes the necessary conditions, that all the roots of an equation may be real. The equation $V=0$ being of the mth degree, it is necessary that the series of signs of the first terms of the functions V, V_1, V_2, &c. should present m variations; this requires that the number of functions should not be less than $m+1$; now, this number cannot surpass $m+1$, since the degree of each function is inferior but one unit to that of the preceding function; therefore it will be $m+1$. It is necessary, besides, that the first terms of the functions, by supposing x positive, should all have the same sign. These conditions are sufficient; for if the exponents of the first terms of the functions only diminish by unity, from each function to the following, they will be alternately even and odd; and if the signs of these first terms only present permanencies when x is positive, they will only present variations for x negative. The number of real roots will then be equal to m.

595. Cor. III. As often as the number of auxiliary functions V_1, V_2, V_3, &c. is equal to m, the number of imaginary roots of the equation $V=0$ can be known by the simple inspection of the signs of the first terms of these functions. The equation $V=0$ has as many pairs of imaginary roots as there are variations in the series of signs of the first terms of the functions V, V_1, V_2, &c. up to the constant V_m inclusively; for if this series present n variations, it presents $m-n$ permanencies. When x is changed into $-x$, the permanencies become variations, and *vice versâ*, so that the number of variations is $m-n$; the number of real roots is therefore $m-2n$.

To find the number of real and imaginary roots in an equation $x^n + ax^{n-1} + bx^{n-2} + \ldots + t = 0$.

596. Denote the equation by V, and its limiting equation by V_1, and perform on these quantities the process of finding their greatest common measure; change the signs of all the terms of the remainders before using them as divisors, and denote these divisors in order by $V_2, V_3, V_4, \ldots, V_n$; the final remainder, with its sign changed V_n, will be independent of x.

Substitute $-\infty$ in the first terms of the quantities V, V_1, V_2, ..., V_n; write down in order the signs of the results, and let the number of changes in these signs from $+$ to $-$, and from $-$ to $+$, be denoted by m; and when $+\infty$ is similarly substituted, let the number of variations be r; then $m-r$ is the number of real roots, and $n-(m-r)$ the number of imaginary roots.

EXAMPLE.

Given the equation $x^4-2x^3-x^2+8x-12=0$ to find the number of its real and imaginary roots.

Here $\quad V = x^4-2x^3-x^2+8x-12,$

and $\quad V_1 = 4x^3-6x^2-2x+8;$

and performing on these quantities the process of finding their greatest common measure, the first remainder is $-5x^2+23x-44$; hence V_2, the next divisor is $5x^2-23x+44$; and the next remainder is found to be $248x-1264$, which, being divided by 8, and the sign changed, gives $V_3 - 31x+158$; and the last remainder is a positive number, or $V_4 = -$; hence

$$
\begin{aligned}
V &= x^4-2x^3-x^2+8x-12,\\
V_1 &= 4x^3-6x^2-2x+8,\\
V_2 &= 5x^2-23x+44,\\
V_3 &= -31x+158,\\
V_4 &= -
\end{aligned}
$$

Since V_4 is a number which does not alter its value by the substitution of any quantity for x, it is unnecessary to write the number, as only the signs of the results are required.

When $-\infty$ and $+\infty$ are substituted in the first terms of these five quantities, the signs of the results are in order of the functions $\quad V, V_1, V_2, V_3, V_4,$

for $x=-\infty$, $+ \; - \; + \; + \; -$ ... 3 variations,

for $x=+\infty$, $+ \; + \; + \; - \; -$ ... 1 variation.

In the former series of signs, the number of variations is *three*, and in the latter only *one*; hence $m=3$, and $r=1$, and $m-r=3-1=2=$ the number of real roots. Hence the number of imaginary roots is $n-(m-r)=4-2=2$.

597. To find the number of roots that lie between any two numbers. Substitute these numbers for x in all the terms of the functions V, V_1, V_2, ..., V_n, and write down in a row the signs of the results; and the difference between the numbers of variations

of signs in these two rows will be the number of roots which lie between the two assumed numbers.

To find the number of roots of the preceding equation that lie between 1 and 3. Substitute these numbers for x, and the signs of the results are

$$\begin{array}{llllllll} & V, & V_1, & V_2, & V_3, & V_4, & & \\ \text{for } x = 1, & - & + & + & + & - & \ldots & \text{2 variations,} \\ \text{for } x = 3, & + & + & + & + & - & \ldots & \text{1 variation.} \end{array}$$

The difference of variations $= 2 - 1 = 1$, or only one root lies between 1 and 3. This root will be found on trial to be 2.

EXERCISES.

1. Find the number of real roots of the equation $8x^3 - 6x - 1 = 0$,

   The three roots are real; there are two roots between 0 and -1, and one between 0 and 1.

2. Find the number of real and imaginary roots of the equation $x^3 - 5x^2 + 8x - 1 = 0$,

   There are two imaginary roots, and one real root between 0 and 1.

3. Find the number of real roots of the equation $x^3 - 2x - 5 = 0$,

   It has two imaginary roots, and one real root between 2 and 3.

4. Find the number of real roots of the equation $x^4 - 2x^3 - 7x^2 + 10x + 10 = 0$,

   It has four real roots; two positive roots between 2 and 3; and two negative roots, one between 0 and -1, and another between -2 and -3.

598. In order to find the next lower integer to the root of an equation, substitute successively for the unknown quantity the numbers 0, 1, 2, 3, 4, ... or 0, -1, -2, -3, ... till two successive results be found with opposite signs; then one root or some odd number of roots lie between the two numbers producing these results; and the smaller of these numbers is therefore the required integer.

599. If it is found that two numbers of the negative series -1, -2, -3, ... produce opposite results, this indicates that at least one negative root lies between them. But if the signs of the alternate terms of the equation be changed, as the second, fourth, sixth, ... the signs of its roots are then changed, and the preceding root will therefore become positive.

Let the equation be $x^3 - 2x - 5 = 0$.

$$\begin{array}{llll} \text{Let } x = 0, \text{ and the result is} & & & = - \ \ 5, \\ \ldots\ x = 1, & \ldots & \ldots & = - \ \ 6, \\ \ldots\ x = 2, & \ldots & \ldots & = - \ \ 1, \\ \ldots\ x = 3, & \ldots & \ldots & = + 16. \\ \ldots & \ldots & \ldots & \end{array}$$

As the results given by substituting 2 and 3 in the equation for x are -1 and $+16$, their opposite signs imply that at least one root lies between 2 and 3; it is therefore $= 2 + y$, where y represents some fraction representing the difference between the true root and the number 2.

NEWTON'S METHOD OF APPROXIMATION.

600. Resuming the equation $f(x) = 0$, if we substitute for x, in this equation, $a + y$ where a is a value very near to the true value of x, suppose that a differs from x by less than $\cdot 1$; by (576) we have

$$f(x) = f(a+y) = f(a) + f'(a)y + \frac{f''(a)}{2}y^2 + \frac{f'''(a)}{2\,.\,3}y^3 + \ldots = 0;$$

$$\therefore\ y = -\frac{f(a)}{f'(a)} - \frac{1}{f'(a)}\left\{f''(a)\frac{y^2}{2} + f'''(a)\frac{y^3}{2\,.\,3} + \ldots\right\}$$

Since a differs from the true root by less than $\cdot 1$, y is less than a tenth, hence y^2 is less than a hundredth, and y^3 is less than a thousandth, &c.; so that by neglecting, in the above value of y, all those terms which contain its powers, we will obtain its value at least to a hundredth part of a unit. And the expression is reduced to

$$y = -\frac{f(a)}{f'(a)}.$$

We effect the division indicated, and stop at the second decimal place; this value being added to a, will give a value of x differing from the true value by less than $\cdot 01$; substituting this more approximate value instead of x, and dividing out to 4 decimal places, a new correction will be found which will, in general, give a value of x true to 4 decimal places; proceeding in the same manner, the third correction will in general be correct to 8 decimal places.

EXAMPLE.

Given the equation $x^3 - 2x - 5 = 0$ to find a value of x.

By the method explained in (598), it is found that the true value differs from 2·1 by less than ·1; hence $a = 2{\cdot}1$ gives

$$y = -\frac{f(2{\cdot}1)}{f'(2{\cdot}1)} = -\frac{(2{\cdot}1)^3 - 2 \times 2{\cdot}1 - 5}{3 \times (2{\cdot}1)^2 - 2} = -\frac{{\cdot}061}{11{\cdot}23} = -{\cdot}0054;$$

$$\therefore a + y = 2{\cdot}1 - {\cdot}0054 = 2{\cdot}0946.$$

Substituting this second value in the above formulary, and calculating the value of y to 8 decimal places, we obtain $y = -{\cdot}00004851$; hence we conclude that $x = 2{\cdot}09455149$ nearly.

Remark. In this example, the assumed value of x was too great; hence the value of y is negative, and it was therefore subtracted from the approximate value; but when the value of y is positive, it must be added to the approximate value.

EXERCISES.

1. Find a root of the equation $x^3 - x^2 + 2x - 3 = 0$,
$\therefore x = 1{\cdot}275682204$.

2. ... ... $x^3 - 15x^2 + 63x - 50 = 0$,
$\therefore x = 1{\cdot}028039$.

3. ... ... $x^3 + 2x^2 - 23x - 70 = 0$,
$\therefore x = 5{\cdot}1346$.

4. ... ... $x^4 - 4x^3 - 3x + 27 = 0$,
$\therefore x = 2{\cdot}2675$.

Another root of the last equation lies between 3 and 4, and is $= 3{\cdot}6797$.

HORNER'S METHOD.

601. This method consists in depressing the given equation to another whose roots shall be less than those of the given equation (570) by the highest digit in one of its roots, and then depressing the depressed equation by the highest digit in its root; and so on continually till a sufficient number of figures be obtained.

After the first depression has been performed, the figure by which it is next to be reduced is found by dividing the absolute term of the depressed equation by the coefficient immediately preceding it, which is the same as the method by which the correction is found in Newton's method, for the absolute term of the depressed equation is $f(a)$, a being the part of the root already found, and the coefficient immediately preceding it is $f'(a)$; when

this next figure is found, the equation is again depressed by this new quantity, by performing the same operation with this new figure on the coefficients of the depressed equation, as was formerly performed with the first figure on the original coefficients; again a new figure is found, and the operation of depression performed; and so on continually till the root is obtained to the required degree of accuracy.

The only difference in the method of performing the operations in Horner's method, and that already exhibited in (570), is that the absolute term is written with its sign changed; that is, it is transposed to the second side, and then the product, which is put under it, is subtracted instead of being added, which gives $-f(a)$ for the remainder.

The first figure of each root must either be found by (582), or by substitution in Sturm's functions, which is the most certain method when all the real roots are required.

EXAMPLE.

1. Find all the real roots of the equation

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$$

The following are Sturm's functions reduced to their simplest form :—

Let $x =$ . . .	$-\infty$	$+\infty$	0	-1	$+1$	$+2$	$+3$	$+5$	6
$V = x^4-8x^3+14x^2+4x-8$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$+$
$V_1 = x^3-6x^2+7x+1$.	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$
$V_2 = 5x^2-17x+6$.	$+$	$+$	$+$	$+$	$-$	$-$	$\mp$	$+$	$+$
$V_3 = 76x-103$ . . .	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$
$V_4 = +$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
Variations . .	4	0	3	4	2	2	1	1	0

Since the variations lost between $-\infty$ and $+\infty$ is $4 - 0 = 4$, the equation has all its roots real. Again, since the variations lost from $-\infty$ to 0 is $4 - 3 = 1$, the equation has *one negative root;* and as the variations for -1 is 4, and for 0 is 3, the negative root lies between 0 and -1. In the same manner, it is found that one variation is lost between 0 and $+1$, another between 2 and 3, and a third between 5 and 6; hence the *positive roots* are between 0 and 1, between 2 and 3, and between 5 and 6.

The equation has therefore three positive roots, and a negative one, which is also manifest (561), by a simple inspection of the signs of the given equation.

In order to find the first figures of the roots that lie between 0 and -1, and between 0 and $+1$, we must narrow the limits by substituting in the above functions $\cdot1, \cdot2, \cdot3, \ldots;\ -\cdot1, -\cdot2, -\cdot3, \ldots,$

till one change is lost by the positive substitution, and one gained by the negative; then the figure immediately preceding that which causes the loss in the positive, or the gain in the negative, will be the first figure of the root. By this means it will be found that the first figure of the *positive* root is ·7, and that the first figure of the *negative* root is also ·7. The same results will be obtained by substituting in the given equation till the results give opposite signs (582).

The root which lies between 5 and 6 is calculated as follows:—

1	− 8	14	4	8 (5·236068
	5	− 15	− 5	− 5
	− 3	− 1	− 1	*13
	5	10	45	10·6576
	2	9	*44 = t	*2·3424
	5	35	9·288	1·93880241
	7	*44	53·288 = d	·40359759
	5	2·44	9·784	·39905490
1 *	12	46·44	*63·072 = t	·00454269
	·2	2·48	1·554747	400954
	12·2	48·92	64·626747 = d	53315
	·2	2·52	1·566321	
	12·4	*51·44	*66·193068 = t	
	·2	·3849	·31608	
	12·6	51·8249	66·50915 = d	
	·2	·3858	·31656	
1 *	12·8	52·2107	66·82571 = t	
	·03	·3867		
	12·83	*52·5974		
	·03	·08		
	12·86	52·68		
	·03	·08		
	12·89	52·76		
	·03			
1 *	12·92			

As the root is carried out only to six decimal places, it is not

necessary to carry the real divisor for the third figure (6) to more than five decimal places; this divisor is 66·50915; and this number, multiplied by ·006, gives eight decimal places, and the dividend ought to be carried to seven or eight decimal places, in order that the figure in the sixth decimal place of the root may be correct. So the divisor 66·825 for the fifth figure of the root requires to be carried only to three decimal places, for the product of this number by ·00006 gives eight decimal places, as it ought to do. So the divisor for the last figure (8) of the root would require to be carried only to two decimal places. The number in the vertical lines preceding the divisors requires to be carried to still fewer places, as the reader will easily perceive from the successive processes of multiplication; and after obtaining the third decimal figure of the root (6), the numbers in the column under b do not require to be multiplied at all.

The root which lies between 2 and 3 is found thus:—

1 − 8	+ 14	+ 4	+ ·8(2·7320508
+ 2	− 12	4	16
− 6	2	8	− 8
2	− 8	− 12	− 7·4599
− 4	− 6	− 4	− ·5401
2	− 4	− 6·657	− 50511759
− 2	− 10	− 10·657	− 3498241
2	·49	− 5·971	− 3411504
0·7	− 9·51	− 16·628	− 86737
7	98	− ·209253	− 85356
1·4	− 8·53	− 16·837253	− 1381
7	1·47	− ·206679	− 1366
2·1	− 7·06	− 17·04393,2	15
7	849	− 1359	
2·83	− 6·9751	− 17·05752	
3	858	− 1359	
2·86	− 6·8893	− 17·0,7,1,11	
3	867		
2·89	− 6·802,6		
3	7		
2·92	− 6·795		

The root whose first figure is + ·7 is found thus:—

1 − 8	+ 14	+ 4	+ 8 (·763932
·7	− 5·11	6·223	7·1561
− 7·3	8·89	10·223	·8439
·7	− 4·62	2·989	·79211376
− 6·6	4·27	13·212	5178624
·7	− 4·13	− 10104	3951341
− 5·9	·14	13·201896	1227283
7	− ·3084	− 28392	1184220
− 5·2	− ·1684	13·173504	43063
·06	− ·3048	− 2368	39472
− 5·14	− ·4732	13·171136	3591
·06	− ·3012	− 2412	2631
− 5·08	− ·7744	13·15872,4	960
·06	− 149	− 72	
− 5·02	− ·789,3	13·1580,0	
·06	− 15	− 7	
− 4·96	− ·804	13·15,7,3	

Since the fourth root is negative, we change the signs of the coefficients of the alternate terms, which changes the positive roots into negative, and the negative into positive (558, Cor.), and then calculate the same as before; thus—

1 + 8	+ 14	− 4	+ 8 (·7320508
·7	6·09	14·063	7·0441
8·7	20·09	10·063	·9559
7	6·58	18·669	·89261841
9·4	26·67	28·732	6328159
7	7·07	1·021947	6171029
10·1	33·74	29·753947	157130
7	·3249	1·031721	154632
10·83	34·0649	30·785668	2498
3	·3258	69478	2473
10·86	34·3907	30·85514,6	25
3	3267	6952	
10·89	34·7174	30·92467	
3	218	173	
10·92	34·739,2	30926,40	
	22	2	
	34·76,1	309,2,8	

602. The proof of the above is very simple when all the roots are calculated, for it was proved (553) that the sum of all the roots of every equation, with their signs changed, is equal to the

coefficient of its second term: it will be found that the sum of the four roots found above fulfil this condition, when we remember that the last root is negative. It is also evident, that after all the roots of an equation except one have been found, it may be obtained by subtracting the sum of those already found from the coefficient of the second term, with its sign changed; but it is better to calculate them all, and then apply this principle as a proof of their accuracy.

603. The student, after carefully studying the preceding example, may easily solve the following exercises, for which one root only is given in the answers, as this is sufficient for illustrating the preceding method; but he should calculate all the roots, and prove their accuracy by (553).

EXERCISES.

1. If $x^3 - 2x - 5 = 0$, . . . $x = 2{\cdot}0945515$.
2. ... $x^3 + 4x^2 - 5x - 20 = 0$, . $x = 2{\cdot}236068$.
3. ... $x^4 - 8x^3 + 20x^2 - 15x + {\cdot}5 = 0$, . $x = 1{\cdot}284724$.
4. ... $x^3 + 10x^2 - 24x - 240 = 0$, . . $x = 4{\cdot}8989795$.
5. ... $x^3 + 12x^2 - 18x - 216 = 0$, . . $x = 4{\cdot}2426407$.
6. ... $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321 = 0$, $x = 8{\cdot}414455$.
7. ... $x^4 - 59x^2 + 840 = 0$, . . . $x = 4{\cdot}8989795$.

604. When one root of an equation is found, the equation may be depressed one degree; that is, if r be one root, and if the equation be divided by $x - r$, there will be no remainder, and the quotient will be an equation one degree lower, the roots of which are the remaining roots of the first equation.

Hence if one root of a cubic equation be r, and the equation be divided by $x - r$, the quotient is a quadratic, the two roots of which are the remaining roots of the cubic equation.

CONTINUED FRACTIONS.

605. A continued fraction is a complex one, whose denominator is an integer with a fraction whose denominator is also an integer with a fraction, and so on.

The most usual form of a continued fraction, having a prefixed integer a, is

$$a + \frac{1}{b} + \frac{1}{c} + \frac{1}{a} +, \text{\&c.}$$

606. The fractions $\frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \ldots$ are called *component* fractions. When the number of component fractions is limited, the continued fraction is called *terminate;* and when the number is indefinite, it is said to be *interminate.*

607. Continued fractions may be generated in various ways. One of the most common is when the value of a quantity, which is not integral, is to be found by a series of approximations.

Let a' be such a quantity; and find the next lower integer to it, which denote by a; then $a' - a \angle 1$; therefore $\frac{1}{a' - a} > 1$, which quantity may be denoted by b'.

If b' be not integral, let b be the next lower integer to it; then $b' - b \angle 1$; therefore $\frac{1}{b' - b} > 1$, which quantity may be denoted by c'.

If c' be not integral, let c be the next lower integer to it, then $c' - c \angle 1$; therefore $\frac{1}{c' - c} > 1$, which quantity may be denoted by d'.

Proceeding in this manner as far as necessary, then

$$\frac{1}{a' - a} = b', \text{ or } a' - a = \frac{1}{b'}, \text{ or } a' = a + \frac{1}{b'},$$

$$\frac{1}{b' - b} = c', \text{ or } b' - b = \frac{1}{c'}, \text{ or } b' = b + \frac{1}{c'},$$

and it would similarly be found that $c' = c + \frac{1}{d'}$,

and that $d' = d + \frac{1}{e'}, \ldots$

Hence by substituting successively the values of b', c', d', ...

$$a' = a + \frac{1}{b'} = a + \frac{1}{b +}\,\frac{1}{c'} = a + \frac{1}{b +}\,\frac{1}{c +}\,\frac{1}{d'},$$

$$\text{or } a' = a + \frac{1}{b +}\,\frac{1}{c +}\,\frac{1}{d +}\,\frac{1}{e +}, \ldots$$

608. Continued fractions may also be derived from common fractions, by performing upon their terms the process of finding their greatest common measure. Thus, if $\frac{A}{B}$ be a fraction, and $a, b, c, d, \ldots$ the quotients obtained by this operation, and $C, D, E, F, \ldots$ the corresponding remainders, then it is evident (103) that

$$A = aB + C,\ B = bC + D,\ C = cD + E, \ldots$$

Hence $$\frac{A}{B} = a + \frac{C}{B},\ \frac{B}{C} = b + \frac{D}{C},\ \frac{C}{D} = c + \frac{E}{D}, \ldots$$

$$\therefore \frac{C}{B} = 1 \div \frac{B}{C} = 1 \div \left(b + \frac{D}{C}\right),$$

and $$\frac{D}{C} = 1 \div \frac{C}{D} = 1 \div \left(c + \frac{E}{D}\right), \ldots$$

Hence substituting successively the values of $\frac{C}{B}, \frac{D}{C}, \frac{E}{D}, \ldots$

$$\frac{A}{B} = a + \frac{C}{B} = a + \frac{1}{b +}\,\frac{D}{C} = a + \frac{1}{b +}\,\frac{1}{c +}\,\frac{E}{D},$$

$$\text{or } \frac{A}{B} = a + \frac{1}{b +}\,\frac{1}{c +}\,\frac{1}{d +}, \ldots$$

609. The quantities $a, b, c, \ldots$ are called *partial* quotients, and the quantities $a + \frac{C}{B}$, $b + \frac{D}{C}$, $c + \frac{E}{D}$, ... are called *complete* quotients. The quantities $a, b, c, \ldots$ here correspond with $a, b, c, \ldots$ in (607); also $\frac{A}{B}, \frac{B}{C}, \frac{C}{D}, \ldots$ correspond with $a', b', c', \ldots$ in that article. If any remainder, as E, be $= 0$, then c is the complete quotient, and the continued fraction terminates with c.

EXAMPLES.

1. Reduce $\frac{25}{11}$ to a continued fraction.

$$\begin{array}{l}
11)25(2 \\
\underline{22} \\
3)11(3 \\
\underline{9} \\
2)3(1 \\
\underline{2} \\
1)2(2 \\
\underline{2}
\end{array}$$

Hence $\frac{25}{11} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2}}}$

610. It is usual to represent a continued fraction by stating the quotients merely; thus, if a' be the given fraction,

$$a' = \frac{25}{11} = 2,\ 3,\ 1,\ 2.$$

2. Reduce $\frac{25}{37}$ to a continued fraction.

As this is a proper fraction, the first quotient is 0; the others are 1, 2, 12; hence

$$\frac{25}{37} = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{12}}}.$$

EXERCISES.

1. Reduce $\frac{37}{15}$ to a continued fraction,

$$a' = 2,\ 2,\ 7, \text{ or } a' = 2 + \cfrac{1}{2 + \cfrac{1}{7}}.$$

2. ... $\frac{13}{29}$ ... ... $a' = 0,\ 2,\ 4,\ 3.$

3. ... $\frac{111}{35}$ ... ... $a' = 3,\ 5,\ 1,\ 5.$

611. Continued fractions may be easily reconverted into common fractions. Thus, taking the first example, the last part

$1+\frac{1}{2}=\frac{3}{2}$; and hence $\frac{1}{1}+\frac{1}{2}=1\div\frac{3}{2}=\frac{2}{3}$; therefore $3+\frac{1}{1}+\frac{1}{2}$ $=3+\frac{2}{3}=\frac{11}{3}$; and $2+\frac{1}{3}+\frac{1}{1}+\frac{1}{2}=2+\frac{1}{\frac{11}{3}}=2+\frac{3}{11}=\frac{25}{11}$.

The other example and exercises may in the same manner be proved to be correct.

612. Let $\frac{U}{V}$ be a fraction, and let the partial quotients of the equivalent continued fraction be a, b, c, d, ... then

$$\frac{U}{V}=a+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+, \ldots$$

Approximate values to the fraction $\frac{U}{V}$, the true value of the whole continued fraction, may be found by converting into a common fraction, the continued fraction carried out to one, two, three, or any number of terms. Let the approximate fraction, found by carrying it inclusively to a, b, c, d, ... p, q, r, ... be respectively denoted by $\frac{A}{A'}, \frac{B}{B'}, \frac{C}{C'}, \frac{D}{D'}, \ldots \frac{P}{P'}, \frac{Q}{Q'}, \frac{R}{R'}, \ldots$ Then is

$$\frac{A}{A'}=\frac{a}{1}, \frac{B}{B'}=a+\frac{1}{b}=\frac{ab+1}{b}, \frac{C}{C'}=a+\frac{1}{b}+\frac{1}{c}=\frac{(ab+1)c+a}{bc+1}, \ldots$$

It is evident that $\frac{C}{C'}$ is $=\frac{Bc+A}{B'c+A'}$, and this expression for $\frac{C}{C'}$ suggests the following theorem in reference to the formation of *convergent* fractions:—

613. The terms of any convergent fraction are equal to the product of the corresponding terms of the preceding convergent, by the partial quotient corresponding to the former, together with the corresponding terms of the next preceding convergent.

Let the theorem be true for $\frac{R}{R'}$ then $\frac{P}{P'}, \frac{Q}{Q'}$, being the two preceding it, it follows that

$$\frac{R}{R'}=\frac{Qr+P}{Q'r+P'}, \text{ and it is to be proved that } \frac{S}{S'}=\frac{Rs+Q}{R's+Q'}.$$

Since $\frac{S}{S'}$ differs from $\frac{R}{R'}$ merely in having $r+\frac{1}{s}$ instead of r, as is

evident by writing down the continued fraction carried out to s; hence

$$\frac{S}{S'} = \frac{Q\left(r+\frac{1}{s}\right)+P}{Q'\left(r+\frac{1}{s}\right)+P'} = \frac{(Qr+P)s+Q}{(Q'r+P')s+Q'} = \frac{Rs+Q}{R's+Q'}.$$

Hence the theorem being admitted to be true for $\frac{R}{R'}$, it is thus proved to be true for $\frac{S}{S'}$; that is, if it is true for one convergent, it is true for the succeeding one. But it was before shewn to be true for $\frac{C}{C'}$ in (612); hence it is true for $\frac{D}{D'}$, and therefore for $\frac{E}{E'}$; and so on—that is, it is generally true.

EXAMPLES.

1. Reduce $\frac{71}{31}$ to a continued fraction, and find the convergents.

$$a' = 2, 3, 2, 4; \text{ hence } \frac{A}{A'} = \frac{2}{1}, \frac{B}{B'} = 2 + \frac{1}{3} = \frac{7}{3},$$

$$\frac{C}{C'} = \frac{2B+A}{2B'+A'} = \frac{2\times 7+2}{2\times 3+1} = \frac{16}{7}, \frac{D}{D'} = \frac{4C+B}{4C'+B'} = \frac{71}{31}.$$

Hence the convergents, arranged under their respective quotients, with an auxiliary fraction $\frac{1}{0}$ prefixed, are

Quotients......... 2, 3, 2, 4,

Convergents.... $\frac{1}{0}, \frac{2}{1}, \frac{7}{3}, \frac{16}{7}, \frac{71}{31}$.

The first fraction $\frac{1}{0}$ is used merely to render the rule applicable to the second convergent $\frac{B}{B'}$, which is here $=\frac{7}{3}$. The fraction $\frac{A}{A'} = \frac{2}{1}$ is always the first quotient. $\frac{B}{B'}$ or $\frac{7}{3}$ is found by taking the quotient over it or 3, multiplying the 2 of $\frac{2}{1}$ by it, and adding the 1 of $\frac{1}{0}$, which gives 7; and then the 1 of $\frac{2}{1}$, multiplied by 3,

and 0 added, gives 3. So to find $\frac{16}{7}$, take the quotient 2 over it; multiply 7 by it, and add 2, which gives 17; then multiply 3 by it, and add 1, which gives 7. And similarly $\frac{71}{31}$ is found.

2. Reduce $\frac{54}{125}$ to a continued fraction, and find the convergents.

Quotients ... 0, 2, 3, 5, 1, 2,

Hence the convergents ... $\frac{1}{0}, \frac{0}{1}, \frac{1}{2}, \frac{3}{7}, \frac{16}{37}, \frac{19}{44}, \frac{54}{125}$.

The first quotient being 0, $\frac{A}{A'} = \frac{0}{1} = 0$.

EXERCISES.

Reduce $\frac{111}{40}, \frac{17}{57}, \frac{1103}{887}$, and $\frac{86400}{20929}$, to continued fractions, and find their convergents.

For $\frac{111}{40}$, Quotients....... 2, 1, 3, 2, 4,

Convergents... $\frac{1}{0}, \frac{2}{1}, \frac{3}{1}, \frac{11}{4}, \frac{25}{9}, \frac{111}{40}$;

for $\frac{17}{57}$, Quotients....... 0, 3, 2, 1, 5,

Convergents... $\frac{1}{0}, \frac{0}{1}, \frac{1}{3}, \frac{2}{7}, \frac{3}{10}, \frac{17}{57}$;

for $\frac{1103}{887}$, Quotients....... 1, 4, 9, 2, 1, 1, 4,

Convergents... $\frac{1}{0}, \frac{1}{1}, \frac{5}{4}, \frac{46}{37}, \frac{97}{78}, \frac{143}{115}, \frac{240}{193}, \frac{1103}{887}$;

$\frac{86400}{20929} = \ldots$ 4, 7, 1, 3, 1, 16, 1, 1, 15,

Convergents... $\frac{1}{0}, \frac{4}{1}, \frac{29}{7}, \frac{33}{8}, \frac{128}{31}, \frac{161}{39}, \frac{2704}{655}, \frac{2865}{694}, \frac{5569}{1349}, \frac{86400}{20929}$.

614. If the numerators and denominators of two consecutive convergent fractions be multiplied in a cross order, the difference of the products will be unity; and for all of these consecutive fractions, the differences between the products are alternately positive and negative.

The difference for $\frac{P}{P'}$ and $\frac{Q}{Q'}$ is $PQ' - P'Q = D_1$,

and for $\frac{Q}{Q'}$ and $\frac{R}{R'}$, observing that (613) $\frac{R}{R'} = \frac{Qr + P}{Q'r + P'}$,

it is $= QR' - Q'R = Q(Q'r + P') - Q'(Qr + P) = P'Q - PQ' = D_2$.

Hence $D_1 = - D_2$, or the differences are equal with opposite signs. The same may be proved for the other convergents. But for $\frac{A}{A'}$ and $\frac{B}{B'}$, or (612) $\frac{a}{1}$ and $\frac{ab+1}{b}$, this difference is ... $= ab - (ab + 1) = - 1$;

hence for $\frac{B}{B'}$ and $\frac{C}{C'}$, it must be $+ 1$; for $\frac{C}{C'}$ and $\frac{D}{D'}$ it is $- 1$; and so on. Hence $PQ' - P'Q$ is either $+ 1$ or $- 1$.

615. Cor. 1. When the first of the products is that of the numerator of the first of two consecutive fractions by the denominator of the following, the difference is $- 1$ when the former fraction is of an odd order, and $+ 1$ when its order is even.

It has been shewn (614) that for $\frac{A}{A'}$ and $\frac{B}{B'}$ the difference is -1; for $\frac{B}{B'}$ and $\frac{C}{C'}$, it is $+ 1$; and so on. Hence if $\frac{P}{P'}$ is of an even order, $PQ' - P'Q = 1$, or $D_1 = 1$, and since (614) $D_1 = - D_2$; hence $D_2 = - 1$.

To illustrate the two last articles, take the third and fourth convergents of Ex. 2 (613), $\frac{3}{7}$ and $\frac{16}{37}$, or $\frac{C}{C'}$ and $\frac{D}{D'}$ and $CD' - C'D = 111 - 112 = - 1$.

616. Cor. 2. The reciprocals of these fractions possess the same property.

For the difference for $\frac{P'}{P}$ and $\frac{Q'}{Q}$ is $P'Q - PQ' = - D_1$, and for $\frac{Q'}{Q}$ and $\frac{R'}{R}$ it is $Q'R - QR' = - D_2$, or D_1, D_2, have here opposite signs to those of D_1 and D_2 in Art. (615).

617. The convergents are in their lowest terms.

For $PQ' - P'Q = \pm 1$, being $+ 1$ when $\frac{P}{P'}$ is of an even order, and $- 1$ when of an odd order.

Now, any quantity that divides P and P' must divide 1; that is, P and P' are relatively prime (96).

618. The corresponding terms of any two consecutive convergents are relatively prime.

For $R = Qr + P$, and if P and Q are relatively prime, so are Q and R. But if Q and R are not prime, let them have a common measure m; then Qr and R being divisible by m, so must P; but it cannot, as P and Q are supposed to be prime; therefore R and Q must be prime.

Hence, when the numerators of the first two of three consecutive convergents are prime, so are the numerators of the two latter. Now, $B = Ab + 1$, and any quantity that divides A and B must divide 1; therefore A and B must be prime. Hence B and C, C and D, ... are prime. The same property is similarly proved to belong to the denominators.

619. The difference between two consecutive convergents is equal to 1 divided by the product of their denominators; and when the two fractions are taken in order, the sign of the difference is positive or negative, according as the first of the two fractions is of an even or an odd order.

For $\dfrac{P}{P'} - \dfrac{Q}{Q'} = \dfrac{PQ' - P'Q}{P'Q'} = \pm \dfrac{1}{P'Q'}$ (614),

taking $+$ or $-$ according as the order of $\dfrac{P}{P'}$ is even or odd.

620. Cor. A convergent is greater or less than the consecutive one, according as the order of the former is even or odd.

To illustrate the two last articles, take the 4th and 5th convergents in the first exercise in Art. (613)—namely, $\dfrac{25}{9}$ and $\dfrac{111}{40}$, or $\dfrac{D}{D'}$ and $\dfrac{E}{E'}$. Then

$$\frac{25}{9} - \frac{111}{40} = \frac{1000 - 999}{9 \times 40} = \frac{1}{360} = \frac{1}{D'E'};$$

and $\dfrac{25}{9}$ or $\dfrac{D}{D'}$ is the greater, its order being the fourth.

621. The convergents are alternately greater and less than the complete fraction, the greater being those of an even order, and the less of an odd order; also, any convergent approaches nearer to the complete fraction than any preceding it.

For $\dfrac{R}{R'} = \dfrac{Qr + P}{Q'r + P'}$, ... and if r' be the complete quotient, corresponding to the partial one r, then r' is equal to the whole of the

continued fraction after r inclusive, or $r' = r + \frac{1}{s + \&c.}$; hence if r' be substituted for r in the preceding expression for $\frac{R}{R'}$, the result must be equal to the complete fraction; hence ... $\frac{U}{V} = \frac{Qr' + P}{Q'r' + P'}$, ... then if $\frac{P}{P'}$ be of an even order, $PQ' - P'Q = +1$ by (615), and

$$\frac{P}{P'} - \frac{U}{V} = \frac{(PQ' - P'Q)r'}{P'(Q'r' + P')} = \frac{r'}{P'(Q'r' + P')} = D',$$

and $$\frac{Q}{Q'} - \frac{U}{V} = \frac{P'Q - PQ'}{Q'(Q'r' + P')} = -\frac{1}{Q'(Q'r' + P')} = D''.$$

Hence D' is positive, and D'' negative; therefore $\frac{U}{V} \angle \frac{P}{P'}$, and $> \frac{Q}{Q'}$; that is, the convergents of an even order are greater, and those of an odd order less, than the complete fraction. Again, since r' is > 1, for r is not $\angle 1$, and $P' \angle Q'$, therefore $D' > D''$, disregarding their signs; that is, the fraction $\frac{Q}{Q'}$ differs less from $\frac{U}{V}$ than $\frac{P}{P'}$ does, and the same is similarly shewn for any convergent and that which precedes it; hence the origin of the term *convergent.*

622. Cor. Any convergent differs from the complete fraction by less than 1 divided by the square of the denominator of the former.

For $r' > 1$, and hence $Q'r' + P' > Q'$; and hence D'' is $\angle \frac{1}{Q'Q'}$ or $\frac{1}{Q'^2}$; that is, $\frac{Q}{Q'}$ differs from $\frac{U}{V}$ by less than $\frac{1}{Q'^2}$; and the same is similarly shewn for any other fractions.

623. To exemplify the two last articles, take the 2d and 3d convergents of Ex. 1, Art. (613)—namely, $\frac{7}{3}$ and $\frac{16}{7}$.

Then $$\frac{U}{V} = \frac{71}{31}, \frac{B}{B'} = \frac{7}{3}, \frac{C}{C'} = \frac{16}{7}, \text{ and}$$

$$D' = \frac{7}{3} - \frac{71}{31} = \frac{4}{93}, \; D'' = \frac{16}{7} - \frac{71}{31} = -\frac{1}{217}.$$

Hence $\frac{7}{3} > \frac{71}{31}$, and $\frac{16}{7} < \frac{71}{31}$. Also $\frac{16}{7}$ differs less from $\frac{71}{31}$ than $\frac{7}{3}$ does. And likewise $\frac{1}{217} < \frac{1}{7^2}$ or $\frac{1}{49}$.

624. Any two consecutive convergents differ from the complete fraction by less than 1, divided by the product of their denominators; and no intermediate fraction can have either of its terms less than either of the corresponding terms of the two convergents.

For $\frac{P}{P'} \sim \frac{Q}{Q'} = \frac{1}{P'Q'} = D$.

But $\frac{U}{V}$ is intermediate between these (621), and hence it must differ from either of them by a quantity less than D.

Again, let $\frac{I}{I'}$ be a fraction intermediate between $\frac{P}{P'}$ and $\frac{Q}{Q'}$, then the difference between $\frac{P}{P'}$ and $\frac{I}{I'}$ cannot be less than D, unless $I' > Q'$, since the numerator of D is 1; neither can the difference between $\frac{Q}{Q'}$ and $\frac{I}{I'}$ be less than D, unless $I' > P'$; therefore I' is greater than either P' or Q'. The same may be proved of their reciprocals; and hence I is greater than either P or Q.

625. The property of convergents proved last is of some practical utility. When, for example, a ratio is expressed by large numbers, and it is required to express the ratio approximately by smaller numbers, one of the convergents will express an approximation in simpler terms than any other fraction.

Let the ratio be that of the circumference of a circle to its diameter carried to 7 decimal places or $\frac{31415926}{10000000}$.

The quotients are 3, 7, 15, 1, 243, &c.

$$\text{Convergents } \frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{86598}{27565}, \text{ \&c.}$$

The second convergent $\frac{22}{7}$ is that found by Archimedes, and the fourth $\frac{355}{113}$ by Adrian Metius. No fraction can express an approximation so near as either of these, unless its terms exceed those of the latter.

626. When one or more of the terms of a continued fraction continually recur, it is called a *periodic* continued fraction.

Let $a + \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{b + , \ldots}}}$ be a periodic fraction. Assume its value $= x$, then $x - a = \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{b + , \ldots}}}$

then since the part of this fraction after the first b is the same as this last fraction, $x - a$ may be substituted for it; hence

$$x - a = \frac{1}{b + x - a}, \text{ or } b(x - a) + (x - a)^2 = 1;$$

$$\therefore x^2 - (2a - b)x = 1 - a^2 + ab,$$

from which the value of $x = \frac{1}{2}(2a - b) \pm \frac{1}{2}\sqrt{(b^2 + 4)}$.

If $a = 1$, and $b = 2$, then $x = \sqrt{2}$, and substituting for a and b in the given fraction, it follows that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + , \ldots}}}$$

This example affords an instance of a continued fraction, the law of whose terms is evident and simple, expressing the value of an irrational number.*

As illustrations of the above example, the student may find the value of x for any other assumed values of a and b.

If the value of $\sqrt{2} = 1{\cdot}4121356, \ldots$ or $\frac{14121356}{10000000}$, be reduced to a continued fraction, it will be found to be the same as the preceding; but by this method the law of the terms could not be logically inferred (142).

* Lord Brounker, the inventor of Continued Fractions, has given an example of one, the law of whose terms is simple, and which expresses the ratio of the area of a circle to the square of its diameter—

namely, as 1 to $1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + , \ldots}}}}$

the numerators of the component fractions being the squares of the odd numbers.

$\frac{1}{\sqrt{2} + 1} = \sqrt{2} - 1$

$$\text{Let } x = \frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+},\ldots$$

then after the first b, x may be substituted for all the terms that follow; hence

$$x = \frac{1}{a+}\,\frac{1}{b+x} = \frac{b+x}{ab+ax+1}.$$

And from this equation is found

$$x = -\frac{b}{2} \pm \frac{1}{2a}\{a(ab^2+4b)\}^{\frac{1}{2}}.$$

Where a and b may have any numerical values.

Let $a = 1$, and $b = 4$, then $x = -2 \pm 2\sqrt{2}$;

therefore $\quad x = -2 \pm 2\sqrt{2} = \frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+},\ldots$

hence $\quad 2\sqrt{2} = 2 + \frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+},\ldots$

EXERCISES. $\sqrt{2} = 1 + (\sqrt{2} - 1$

1. Find the value of $x = \frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+},\ldots$ $\quad = 1 + \frac{1}{\sqrt{2}+}$

$= \ldots\; x^2 + ax - 1 = 0$, and $x = -\frac{a}{2} \pm \frac{1}{2}\sqrt{(a^2+4)}$.

2. ... ... $x = a + \frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{b+}\,\frac{1}{c+},\ldots$

The equation is $bx^2 - (2ab - bc)x + a^2b - abc - c = 0$, from which the value of x can easily be found.

$\sqrt{2} - 1 = \frac{1}{\sqrt{2}+1}$

$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \frac{1}{2 + \frac{1}{\sqrt{2}+}}$

$= 1 + \frac{1}{\ldots}$

INDETERMINATE EQUATIONS.

627. Equations are said to be *indeterminate* when the number of unknown quantities exceeds that of the equations.

In indeterminate equations the number of values of the unknown quantities is indefinite; but when their values are restricted by certain conditions, the number of them is generally limited. The conditions usually imposed are, that the values be integral numbers; or, by still further restrictions, that they be also positive, or that they be square numbers, or cube numbers, and so on.

I.—SIMPLE INDETERMINATE EQUATIONS.

PROBLEMS.

I. Given the equation $ax + by = c$, to find all the positive and integral values of x and y; b and c being either positive or negative.

Convert $\frac{a}{b}$ into a continued fraction, and find the last convergent, which may be denoted by $\frac{m}{n}$; then (614) $an \sim bm = 1$. If bm be the greater, find the value of x,—namely, $\frac{c - by}{a}$ from the given equation, and multiply its numerator by m; it then becomes $\frac{mc - bmy}{a}$, which may be reduced by division to the form $q + \frac{d - y}{a}$. Assume $\frac{d - y}{a} = r$, and then $y = d - ar$; then substitute this value for y in the preceding value of x, and its value will be of the form $x = e + br$. By giving r proper integral values, the positive integral values of x and y will be found.

If an were the greater, then a value of y would be found, and a similar process performed.

When the system of least values of x and y are known, the other systems may be found by adding the coefficients of y to the values of x, and the coefficients of x to the values of y, when the equation is $ax - by = \pm c$. When the equation is $ax + by = \pm c$, and a system of values is got, such that the value of x is the greatest; then the other values of x and y are obtained by subtracting from the values of x the coefficient of y, and adding to the values of y the coefficient of x.

That is, if m and n be a system of values of x and y, then the other values of x and y, for the equation $ax - by = \pm c$, are

$$x = m + br,\ y = n + ar,$$

and for the equation $ax + by = \pm c$, they are

$$x = m - br,\ y = n + ar,$$

where r may have any proper positive integral values.

EXAMPLES.

1. Given $15x - 13y = 10$, to find the positive integral values of x and y.

By converting $\frac{15}{13}$ into a continued fraction, the last convergent is found to be $\frac{7}{6}$; hence $\frac{a}{b} = \frac{15}{13}, \frac{m}{n} = \frac{7}{6}$, and

$$an - bm = 15 \times 6 - 13 \times 7 = 90 - 91 = -1,$$

as bm is the greater, find a value of x; then $x = \frac{10 + 13y}{15}$; hence multiplying by $m = 7$, $\frac{7(10 + 13y)}{15} = \frac{70 + 91y}{15} = 4 + 6y + \frac{10 + y}{15}$, therefore assume $\frac{10 + y}{15} = r$; hence $y = 15r - 10$, and $x = \frac{10 + 13y}{15} = \frac{10 + 13 \times 15r - 130}{15} = 13r - 8$.

If r were assumed $= 0$ or $=$ a negative number, the values of x and y would be negative; hence r must be assumed > 0, or 1 is its least value; but its value may be assumed equal to any greater number than 1; the number of solutions is therefore indefinitely great.

When $\quad r = 1,\ x = 5,\ y = 5,$

$$r = 2,\ x = 18,\ y = 20,$$

$$r = 3,\ x = 31,\ y = 35,$$

$$\cdots \quad \cdots \quad \cdots$$

628. When the least values $x = 5, y = 5$, are found, the other values are found; thus (627),

$$x = p + br,\ y = q + ar,$$

$$\text{or } x = 5 + 13r,\ y = 5 + 15r.$$

When $\qquad r = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6, \ldots$

then $\qquad x = 5, 18, 31, 44, 57, 70, 83, \ldots$

and $\qquad y = 5, 20, 35, 50, 65, 80, 95, \ldots$

So that the successive values of x and y are found by adding respectively to their least values the coefficients of y and x; and the two series of values form equidifferent progressions, the common difference of the former being 13, and of the latter 15.

2. Given $10x + 7y = 763$, to find the values of x and y.

Here $\frac{a}{b} = \frac{10}{7}, \frac{m}{n} = \frac{3}{2}, an \sim bm = 20 \sim 21 = 1$;

hence as bm is the greater, find a value of x,

$$x = \frac{763 - 7y}{10} = 76 + \frac{3 - 7y}{10}.$$

The 76 may be neglected, as $\frac{3 - 7y}{10}$ must be a whole number, since x and 76 are so. Hence as $m = 3$,

$$\frac{3(3 - 7y)}{10} = \frac{9 - 21y}{10} = -2y + \frac{9 - y}{10}.$$

Let $\frac{9 - y}{10} = r$, then $y = 9 - 10r$,

and $\qquad x = \frac{763 - 7y}{10} = \frac{763 - 63 + 7 \times 10r}{10} = 70 + 7r.$

If $r = -10$, then x would $= 0$, and if $r = 1$, y would $= -1$; the limits of r therefore are 1 and -10; so that r can have only the integral values intermediate between 1 and -10; namely, $0, -1, -2, \ldots$ to -9 inclusive, or 10 values; so that there are only 10 positive integral systems of values of x and y.

When $\qquad r = \ \ 0, x = 70, y = \ \ 9,$

$\qquad r = -1, x = 63, y = 19,$

$\qquad r = -2, x = 56, y = 29,$

$\qquad \ldots \qquad \ldots \qquad \ldots$

629. Or the values are found thus (627), when the system $x = 70$, $y = 9$, are known,

$$x = p - br,\ y = q + ar,$$

or $$x = 70 - 7r,\ y = 9 + 10r.$$

The limits of r are evidently 0 and 9.

When $\quad r = 0, 1, 2, 3, \ldots 9,$

then $\quad x = 70, 63, 56, 49, \ldots 7,$

and $\quad y = 9, 19, 29, 39, \ldots 99.$

3. Let $3x - 11y = 20$, to find x and y.

Here $\frac{a}{b} = \frac{3}{11}, \frac{m}{n} = \frac{1}{4}$, $an - bm = 12 - 11 = 1$, and as an is the greater, find a value of y, and multiply it by $n = 4$ $y = \frac{3x - 20}{11}, \frac{4(3x - 20)}{11} = \frac{12x - 80}{11} = x - 7 + \frac{x - 3}{11}$.

Let $\frac{x-3}{11} = r$, then $x = 11r + 3$,

and $$y = \frac{3x - 20}{11} = \frac{33r + 9 - 20}{11} = 3r - 1.$$

If $r = 0$, $y = -1$; hence for positive values, r must be at least $= 1$.

When $r = 1$, $x = 14$, and $y = 2$.

By making $r = 2, 3, 4, 5, \ldots$ an indefinite number of values may be found.

630. Sometimes the value of x may be reduced to the form $\frac{d - y}{a}$, or $\frac{y}{a}$, without requiring to multiply; and in such a case, $\frac{d-y}{a}$, or $\frac{y}{a}$, may be assumed $= r$, as in the following example:—

4. In how many different ways can £100 be paid in crowns and guineas; and how may it be paid in this manner so as to require the smallest number of crowns?

Let $\quad x =$ the number of crowns,

and $\quad y = \ldots \quad \ldots$ guineas,

then $$5x + 21y = 2000;$$

$$\therefore x = \frac{2000 - 21y}{5} = 400 - 4y - \frac{y}{5}.$$

Here the coefficient of y in $\frac{y}{5}$ is 1; hence let $\frac{y}{5} = r$, then $y = 5r$, and $x = 400 - 21r$.

The least value of x will evidently be found by giving r its greatest admissible value; that is, $r = 19$; then $x = 1$, and $y = 95$; so that £100 may be paid by 95 guineas and a crown.

The least admissible value of r is evidently 1; for if it be less, then y would be 0 or negative. Hence the whole number of ways required is found by giving r all the integral values from 1 to 19 inclusive; that is, 19 different ways.

631. This rule depends on the following principles:—

1. Any integral multiple of an integral quantity is also an integral quantity.

Hence if $x = \dfrac{c \pm by}{a}$ is an integer, and so is $\dfrac{m(c \pm by)}{a}$.

2. If the sum or difference of two quantities be integral, and if one of them be an integer, so is the other.

Hence if $\dfrac{m(c \pm by)}{a} = q \pm \dfrac{d \pm y}{a}$ be an integral quantity; so is $\pm \dfrac{d \pm y}{a}$. And if this quantity be assumed $= r$, then $y = \pm ar \mp d$; and any arbitrary integral values being assigned to r, the result will be an integral value of y. Also $y \pm d = \pm ar$, and $y \pm d$ is therefore a multiple of r by a; hence $\dfrac{d \pm y}{a}$ is equal to an integer, and hence also $q \pm \dfrac{d \pm y}{a}$ is integral, and therefore its equal $\dfrac{m(c \pm by)}{a}$ is also integral; but a and m are relatively prime (618); hence $c \pm by$ is a multiple of a (150); and therefore x is also an integral number.

632. The formulas given in articles (628) and (629) may be proved in this manner:—

Let the equation be $ax - by = \pm c$, and let m and n be a system of values of x and y, and x', y', any system whatever, then

$$am - bn = \pm c = ax' - by';$$

hence $$a(x' - m) = b(y' - n), \text{ and } \frac{x' - m}{y' - n} = \frac{b}{a} = \frac{br}{ar},$$

where br and ar may be both positive or both negative. Since in the preceding fraction x' and y' denote any system of values, $x' - m$, $y' - n$, will not be respectively equal to b and a, except for one system of values; hence they must be considered equal to a fraction, whose terms are multiples of b and a—namely, br and ar, r being either positive or negative.

Therefore $x' = m + br$, $y' = n + ar$.

Here a, b, m, and n, are known; and by giving r proper integral values, those of x and y are known.

When $r = 0, \quad 1, \quad 2, \quad 3, \ldots \quad r,$

$$x = m,\ m + b,\ m + 2b,\ m + 3b,\ \ldots\ m + br,$$

$$y = n,\ n + a,\ n + 2a,\ n + 3a,\ \ldots\ n + ar.$$

The values of r may also be taken negatively.

633. Therefore the values of x and y are equidifferent series, having respectively the common differences b and a.

When the equation is $ax + by = \pm c$, it may be similarly proved that

$$x = m - br,\ y = n + ar,$$

where r may also be positive or negative.

634. The equation $ax \pm by = \pm c$ is always impossible in integers, when a and b have any measure that is not also a measure of c.

For let m be a measure of a and b, but not of c, and let $a = a'm$, $b = b'm$, then

$$a'x \pm b'y = \pm \frac{c}{m}.$$

And if x and y have any integral values assigned to them, the first member of this equation would be integral, and the second fractional, which is impossible. Hence,

635. In order that x and y may have integral values, a and b must be relatively prime; or, if not, their greatest common measure must measure c.

636. In order to determine when the equation $ax \pm by = \pm c$ is possible in positive integers, let $\frac{m}{n}$ be the convergent fraction nearest in value to $\frac{a}{b}$. Then $an - bm = \pm 1$, taking $+1$ when the convergent is of an odd order, and -1 when its order is even (621).

1. For the equation $ax - by = \pm c$.

1*st*, Let the convergent be of an odd order, then

$$an - bm = 1;\ \text{hence}\ a \,.\, cn - b \,.\, cm = c.$$

Therefore one solution of $ax - by = c$ is $x = cn$, $y = cm$; and hence its other solutions are determined by the formulas in Art. (632), which become here

$$x = cn + br,\ y = cm + ar.$$

And for positive values of r, the values of x and y are all positive.

2*d*, Let the convergent be of an even order; then

$$an - bm = -1, \text{ hence } a(-cn) - b(-cm) = c.$$

And therefore one solution of $ax - by = c$, is $x = -cn$, $y = -cm$, and the other solutions are contained in the formulas in Art. (632), which here become

$$x = -cn + br, \; y = -cm + ar,$$

and it is evidently possible to give r such positive values as will make the values of x and y positive.

In the same manner it may be proved that the equation $ax - by = -c$ always admits of positive integral values of x and y, by multiplying by c, when $an - bm = -1$, and by $-c$ when $an - bm = +1$. Hence

637. The equation $ax - by = \pm c$ is always possible in positive integers when a, b, and c, fulfil the conditions in Art. (635), and the number of solutions is unlimited.

2. For the equation $ax + by = \pm c$.

1*st*, Let the convergent be of an odd order; then

$$an - bm = 1, \text{ and hence } an + b(-m) = 1;$$

therefore $$a \,.\, cn + b(-cm) = c.$$

One solution of $ax + by = c$ therefore is $x = cn$, $y = -cm$, and its other solutions are (633).

$$x = cn - br, \; y = -cm + ar.$$

638. And if such a value can be assigned to r that $cn - br$ may be positive, and also $-cm + ar$, the resulting values of x and y would be positive.

The limits of the values of r will shew the number of solutions.

2*d*, Let $an - bm = -1$, then $an + b(-m) = -1$,

and $$a(-cn) + b \,.\, cm = c.$$

From which $x = -cn$, $y = cm$, and the conclusion is similar to the preceding, observing that the values of r may be positive or negative.

639. Exactly similar conclusions are obtained for the equation $ax + by = \pm c$, by multiplying by $-c$ when $an + b(-m) = +1$; and by $+c$ when $an + b(-m) = -1$.

640. It can also be proved that the equation $ax + by = c$ admits of positive integral values of x and y when a and b are relatively prime, and when $c > \{ab - (a + b)\}$; but though this condition is sufficient, it is not always necessary.

641. When the coefficient of one of the unknown quantities is 1, the equation is easily solved, by assigning to the other any integral positive values.

642. When in the equation $ax \pm by = \pm c$ one of the coefficients, as b, is a submultiple of c, as $c = bm$, then $y = \pm m - \frac{ax}{b}$, and when $x = 0$, $y = \pm m$; and by means of this solution the rest may be easily obtained.

643. When the equation is $ax \pm by = 0$, one solution is evidently $x = 0$, $y = 0$; the rest are $x = br$, $y = \mp ar$.

EXERCISES.

644. In the following exercises, the required values of x and y are positive and integral:—

1. Given $7x - 12y = 15$, to find the least values of x and y, and also the number of solutions,
$x = 9$, $y = 4$, and the number of solutions unlimited.

2. Find the least values of x and y in the equation $20x - 31y = 7$, ... $x = 5$, and $y = 3$.

3. Given $24x - 13y = 16$, to find a value of x and y,
The least values are $x = 5$, and $y = 8$.

4. ... $25x - 16y = 12$, to find x and y,
The least values are $x = 12$, and $y = 18$.

5. ... $8x - 3y = 16$, to find x and y,
$x = 5, 8, 11, 14, \ldots$
$y = 8, 16, 24, 32, \ldots$

6. ... $7x + 11y = 47$, to find x and y,
Only one solution: $x = 2$, and $y = 3$.

7. ... $5x + 7y = 19$, to find x and y,
Only one solution: $x = 1$, and $y = 2$.

8. ... $5x + 24y = 109$, to find x and y,
Only one solution: $x = 17$, and $y = 1$.

9. ... $29x + 17y = 250$, to find x and y,
Only one solution: $x = 1$, and $y = 13$.

10. ... $17x - 49y = -8$, to find x and y,
The least values, $x = 37$, and $y = 13$.

11. Given $27x + 16y = 1600$, to find x and y,

There are only three solutions: $\begin{cases} x = 48, 32, 16. \\ y = 19, 46, 73. \end{cases}$

Apply to this example the remark in (642).

12. A owes B £100, but A has no money but guineas, and B has only 40 crowns: how can the debt be paid, and in how many ways? A gives B 100 guineas, and receives 20 crowns from him. This is the only solution.

13. A owes B a shilling, but A has only guineas, and B has only louis-d'ors worth 17s. each: how can A pay the debt with the smallest number of guineas?

A gives B 13 guineas, and receives from him 16 louis-d'ors.

14. A number of sheep and oxen were bought for £128, 10s.; the sheep at 17s., and the oxen at £10 a head: how many were bought? . . . The only solution is, 12 oxen and 10 sheep.

15. How can 21 guineas be paid in sovereigns and pistoles of 17s.: and in how many ways can it be done?

Only in one way—namely, by 11 sovereigns and 13 pistoles.

16. Is it possible to pay £18, 19s. in guineas and sovereigns?

It is not possible.

17. Can £30, 17s. be paid in guineas and sovereigns; and if so, in how many different ways can it be done?

The only way—with 13 sovereigns and 17 guineas.

18. A merchant wishes to mix two kinds of tea at 4s. 6d. and 5s. 4d. per lb., so as to make a mixture worth 5s. per lb.: how much must he use of each, the quantities being whole numbers?

2 lbs. at 4s. 6d. and 3 lbs. at 5s. 4d., which is the least solution. The number of solutions is unlimited.

19. How can 78 francs be paid with pieces of 5 francs and of 3 francs, and in how many ways?

With 3 pieces of 5 francs, and 21 of 3 francs. There are five solutions.

II. Given two equations containing three unknown quantities, to find positive integral values of them.

645. Let x, y, and z, be the three quantities; eliminate one of them, as z, and a new indeterminate equation will result, containing only x and y; find expressions for their integral values in terms of a quantity r, as in the preceding problem; then substitute these values of x and y in either of the given equations, and a new equation will arise, containing only r and z, which will be divisible by the coefficient of z, and will thus afford an integral value of z

in terms of r. Substitute this value of z in the values of x and y; and the values of x, y, and z, being now expressed in an integral form in terms of r, their systems of values will be easily found by assigning to r proper values.

646. Sometimes the value of z cannot be expressed in terms of r in an integral form, and then the equation in terms of z and r must be solved, as in the preceding problem; and if s be the new quantity to be assumed, the values of z and r can be expressed in terms of s in an integral form, and then those of x and y can be expressed in the same form in terms of s; and then by assigning to s proper values, the systems of values of x, y, and z, may be found.

This exception to the general rule occurs when the equation in terms of x and y has some divisor, though the exception does not always exist when this is the case.

EXAMPLES.

1. Find the integral positive values of x, y, and z, in the equations

$$10x - 7y + 5z = 85,$$

$$9x + 5y - 8z = 360.$$

Multiply the former equation by 8, and the latter by 5, and add the products, and there results the equation

$$125x - 31y = 2480.$$

Find a value of y, and proceed in solving this equation as in the former problem, and there is found

$$x = 31r, y = 125r - 80.$$

Substitute these values of x and y in either of the given equations, as the second, and after collecting the terms, there results

$$904r - 8z = 760,$$

or $$z = 113r - 95.$$

When $$r = 1, \quad 2, \quad 3, \ldots$$

$$x = 31, \quad 62, \quad 93, \ldots$$

$$y = 45, 170, 295, \ldots$$

$$z = 18, 131, 244, \ldots$$

The number of values is unlimited. The values of x and y increase by the common differences 31 and 125, the coefficients of the above equation in terms of x and y, and that of z increases by 113.

2. Find the positive integral values of x, y, and z, in the equations

$$6x + 7y + 4z = 122,$$
$$11x + 8y - 6z = 145.$$

Eliminate z, as in the preceding example, and there results,

$$80x + 74y = 1312, \text{ or } 40x + 37y = 656.$$

Solve this equation by the first problem, and there is found

$$x = 9 - 37r,\ y = 8 + 40r.$$

Substitute these values of x and y in the first equation, then

$$4z + 58r = 12, \text{ or } 2z + 29r = 6,$$

and $\qquad z = 14r + 3 + \dfrac{r}{2}.\quad$ Let $\dfrac{r}{2} = s$, then $r = 2s$.

Hence $z = 3 + 29s$, then $y = 8 + 80s$, and $x = 9 - 74s$; when $s = 0$, $x = 9$, $y = 8$, $z = 3$.

This is the only possible system of values.

In this example the equation in terms of x and y is divisible by 2; and hence it was necessary to assume a new quantity s, in order that integral values of z might be obtained (646).

EXERCISES.

1. Find the positive integral values of x, y, and z, and the number of solutions, in the two equations

$$4x - y + 5z = 127,$$
$$2x - 3y + 7z = 131.$$

There are only three solutions—namely,

$$x = 1,\ 5,\ 9,$$
$$y = 27,\ 18,\ 9,$$
$$z = 30,\ 25,\ 20.$$

2. Find the positive solutions of the two equations

$$5x - y + 4z = 127,$$
$$7x - 3y + 2z = 151.$$

There are only two solutions—namely,

$$x = 20,\ 25,$$
$$y = 1,\ 10,$$
$$z = 7,\ 3.$$

3. Find the positive solutions of the equations

$$4x - 3y + 6z = 157,$$

$$9x - 8y + 10z = 311.$$

There are only three solutions,

$$x = 25, 43, 61,$$

$$y = 3, 17, 31,$$

$$z = 11, 6, 1.$$

647. To demonstrate the rule, let the given equations be

$$ax + by + cz = d \quad \ldots \quad [1],$$

$$a'x + b'y + c'z = d' \quad \ldots \quad [2].$$

In order to eliminate z, multiply the former by c', and the latter by c, and take the difference of the products; then

$$(ac' - a'c)x + (bc' - b'c)y = c'd - cd' \quad \ldots \quad [3].$$

Hence if x', y', are a system of values of x and y in this equation, their other values are expressed by (633),

$$x = x' + (bc' - b'c)r,\ y = y' - (ac' - a'c)r;$$

and when these values are substituted for x and y in either of the given equations, as in the first, there results

$$a(bc' - b'c)r - b(ac' - a'c)r + cz = e,$$

where e represents all the other terms, which are numbers,

or $$(a'b - ab')cr + cz = e \quad \ldots \quad [4].$$

And as c is a divisor of the first member, it must also divide the second, if the problem be possible; hence if $e = ce'$,

$$z = (ab' - a'b)r + e'.$$

And thus the value of z is expressed in an integral form, in terms of r, as those of x and y are.

When the equation [3] has a divisor, it must be suppressed, and then c is not necessarily a factor of the coefficient of r in [4]; and hence this equation may not be divisible by c, and if not, it will be of the form

$$cz + kr = e,$$

from which the values of z and r must be found in an integral form in terms of a new quantity s, by solving it for the unknown quantities z and r, by the rule of the preceding problem. When r is found in terms of s, then z, and hence also x and y, can be found in terms of s; and by assigning to it proper values, those of x, y, and z, may be found.

III. Given one equation only, and three unknown quantities, to find their positive integral values.

648. When the number of unknown quantities exceeds that of the equations by more than one, the equations are said to be *more than* indeterminate; for one or more of the unknown quantities may then have arbitrary values assigned, and the equation or equations will still be indeterminate.

649. Let the equation be $ax + by + cz = d$;

then if $\qquad d - cz = c', \; ax + by = c'.$

Find now from this equation the values of x and y in terms of a quantity r, as in Problem I., then the expressions for x and y will also contain z, and by assigning any proper values to r and z, those of x and y will be found.

650. Limits to the value of z may easily be found by considering that its value cannot be $\angle 1$; and as its value is greatest when those of x and y are least, and their least values cannot be $\angle 1$, the greatest value of z cannot exceed $z = \dfrac{d - a - b}{c}$; for when x and y are 1, the given equation becomes $a + b + cz = d$. The least value of z, however, may exceed 1, and its greatest may be less than the above quantity.

651. The number of solutions will evidently be limited by the extreme values of z, and may easily be found when these values are known. Let z' and z'' be the greatest and least values of z, then the number of solutions will be $= z'' - z' + 1$.

652. When two of the terms of the equation containing two of the unknown quantities have opposite signs, and their coefficients fulfil the conditions in Art. (635), the equation admits of an unlimited number of solutions.

For let the equation be $ax - by + cz = d$, or $ax - by = c'$, then for any particular value of z, this equation admits of an infinite number of solutions (637), if its coefficients fulfil the conditions in article (635.)

EXAMPLE.

Let $5x + 6y + 20z = 187$;

then $\qquad 5x + \; 6y = 187 - 20z = c'$,

and $x = \dfrac{c' - 6y}{5} = -y + \dfrac{c' - y}{5}$. Let $\dfrac{c' - y}{5} = r$, then $y = c' - 5r = 187 - 20z - 5r$; hence $x = -187 + 20z + 6r$. The limits of z are (650) 1, and $\dfrac{187 - 5 - 6}{20} = \dfrac{176}{20} = 8\frac{4}{5}$.

When $z = 1$, $x = -167 + 6r$, $y = 167 - 5r$,
and the limits of r are then evidently 28 and 33; and it may, therefore, have $33 - 28 + 1 = 5 + 1 = 6$ values. Hence, when

$$z = 1, \text{ and } r = 28, 29, 30, 31, 32, 33,$$

$$\begin{cases} x = 1, 7, 13, 19, 25, 31, \\ y = 27, 22, 17, 12, 7, 2, \end{cases}$$

the values of x increasing by 6, and those of y diminishing by 5.

It is similarly shewn that when $z = 2$, the limits of r are 25 and 29, and the number of solutions five; the first being $x = 3$, $y = 22$, and the last $x = 27$, $y = 2$.

So when $z = 3$, there are four solutions; when $z = 4$, there are four; when $z = 5$, there are three; for $z = 6$, there are two; for $z = 7$, two; and for $z = 8$, there is only one. The reader may find these solutions as an exercise.

When $z = 8$, then $x = 3$, $y = 2$. And $8\frac{4}{5}$ being the limit of z, there can be no solution for greater values of z. This appears from the values of x and y; for when $z = 9$, $x = -7 + 6r$, $y = 7 - 5r$, and no positive value of r can give x and y positive integral values.

653. In the above equation x cannot have an even value, for then $6y + 20z$ would be equal to an odd number, while 2 is a divisor of 6 and 20. Also, since $5x + 20z = 187 - 6y$, and the first member is divisible by 5, the second must be so too. Hence $6y$ must have 2 in the units' place; that is, y must have either 2 or 7 in its units' place. The values of z are not thus restricted, for 5 and 6 are prime to each other. It is therefore in this example more convenient to transpose z to the second member, as its values may be consecutive. The remarks in this article could all have been made *à priori*, or before solving the equation. It would be tedious to state all the relations that may be observed.

654. In the equation $ax + by = d - cz = c'$, if a and b are relatively prime, then for any integral values of z the equation is possible in integers, either positive or negative (635); and if it affords positive integral solutions for two such values of z, it will do so for all intermediate values, so that the values of z may be consecutive. When, however, a and b have a measure m, that does not divide d and c or c', the equation is then possible only for such values of z as make c' a multiple of m; so that in this case consecutive values cannot be assigned to z. In this case, therefore, instead of taking cz to the second member, it would be preferable to take ax, provided b and c are relatively prime. In the second of the following exercises the term $3y$ ought not to be transposed, as 10 and 2 are not prime to each other.

EXERCISES.

1. Given $5x + 4y + 3z = 24$, to find the positive integral values of x, y, and z.

When $z = 1, x = 1, y = 4.$
... $z = 2, x = 2, y = 2.$
... $z = 5, x = 1, y = 1.$

There are only three solutions.

2. Find the positive and integral values of x, y, and z, in the equation $10x + 3y + 2z = 29$,

$$\text{When } x = 1, \begin{cases} y = 5,\ 3,\ 1, \\ z = 2,\ 5,\ 8, \end{cases} \text{ when } x = 2, \begin{cases} y = 1. \\ z = 3. \end{cases}$$

There are only four solutions.

IV. To find the least integral and positive numbers that, being divided by given divisors, shall leave given remainders.

655. Let the required number be denoted by V; let x, x', x'', x''', ... be the unknown quotients; d, d', d'', d''', ... the given divisors; and r, r', r'', r''', ... the given remainders; then is

$$V = dx + r = d'x' + r' = d''x'' + r'' = d'''x''' + r''' = \ldots \quad [1].$$

Let the equation

$$dx + r = d'x' + r', \text{ or } dx - d'x' = r' - r \ \ldots \quad [2].$$

be solved for the positive integral values of x and x' by Problem I.; and let x and x' be expressed in terms of a new quantity p, assumed as r was in that problem; then the values of x and x' will be of the form

$$x = ap + b,\ x' = a'p + b'; \text{ and hence if } c' = d'b' + r'$$

$$V = d'x' + r' = a'd'p + c' \ \ldots \quad [3].$$

The equation $d'x' + r' = d''x'' + r''$ now becomes

$$a'd'p + c' = d''x'' + r'', \text{ or } a'd'p - d''x'' = r'' - c' \ \ldots \quad [4],$$

and is to be solved for the positive and integral values of p and x'' by Problem I., and then p and x'' will be expressed in terms of some new quantity p'; thus,

$$p = mp' + n,\ x'' = a''p' + b''; \text{ and hence if } c'' = d''b'' + r''$$

$$V = d''x'' + r'' = a''d''p' + c'' \ \ldots \quad [5].$$

The equation $d''x'' + r'' = d'''x''' + r'''$ now becomes

$a''d''p' + c'' = d'''x''' + r'''$, or $a''d''p' - d'''x''' = r''' - c''$ ... [6],

and is to be solved as before for p' and x''', which will be expressed in terms of some new quantity p''; thus,

$$p' = m'p'' + n',\ x''' = a'''p'' + b''';\ \text{and hence}$$
$$V = d'''x''' + r''' = a'''d'''p'' + c''' \ \ldots \qquad [7].$$

If there are no more conditions than the above four to be fulfilled, assume now the least value of p'' that will make V positive; and this will be the required number.

656. Another method of solving the above problem is as follows:—

Let $d_1, d_2, d_3, \ldots d_n$, be the divisors, $r_1, r_2, r_3, \ldots r_n$, the remainders, $N_1, N_2, N_3, \ldots N_n$, the numbers which fulfil the first, second, third, ... nth conditions; also let L_2 be the least common multiple of d_1, d_2, L_3, that of d_1, d_2, d_3, and so on to L_n, the least common multiple of the first n divisors.

Now, it is plain that the least number which fulfils the first condition is $d_1 + r_1$; hence $N_1 = d_1 + r_1$; and any number of ds may be added to this without altering the remainder; hence $N_2 = pd_1 + N_1$; and as this, when divided by d_2, gives a remainder r_2, we must have $\dfrac{pd_1 + N_1 - r_2}{d_2}$ = an integer, and the least value of p that fulfils this condition will determine N_2. In the same manner, the remainders will not be affected by adding to N_2 any number of times L_2, since L_2 divides by d_1 and d_2 without remainder; so that $N_3 = pL_2 + N_2$; and as this, divided by d_3, leaves a remainder r_3, we have $\dfrac{pL_2 + N_2 - r_3}{d_3}$ = an integer, and the least value of p that fulfils this condition gives N_3. In the same manner we proceed to find N_4, N_5, &c. the general formula being $\dfrac{pL_{n-1} + N_{n-1} - r_n}{d_n}$ = an integer, which determines p, and then we have $N_n = pL_{n-1} + N_{n-1}$; and as we have taken at each step the least number that would fulfil the previous conditions, we have at the final step the least number that fulfils all the conditions.

EXAMPLES.

1. Find the least number that, being divided by 3, 5, and 7, shall leave the remainders 2, 1, and 6.

Let x, y, and z, be the quotients, then

$$V = 3x + 2 = 5y + 1 = 7z + 6$$

$$3x + 2 = 5y + 1, \text{ and } x = \frac{5y-1}{3}, \text{ and } \frac{5(5y-1)}{3}$$

$$= \frac{25y-5}{3} = 8y - 1 + \frac{y-2}{3}.$$

Let $\frac{y-2}{3} = r$, then $y = 3r + 2$.

Hence $5y + 1 = 7z + 6$ becomes $15r + 11 = 7z + 6$, and

$$z = \frac{15r+5}{7} = 2r + \frac{r+5}{7},$$

and if $\frac{r+5}{7} = s$, $r = 7s - 5$, and $z = 15s - 10$,

and $$V = 7z + 6 = 105s - 64.$$

The least value of V is when $s = 1$, then $V = 41$.

So that 41 is the least number that satisfies the three conditions of the question.

657. The remainders may be negative. Thus, 17, divided by 5, gives three for a quotient, and 2 for a remainder, or 4 for a quotient and -3 for a remainder; so that

$$17 = 3 \times 5 + 2, \text{ or } = 4 \times 5 - 3.$$

2. Find the least number that, being divided by 3, 7, and 9, shall leave 1, 4, and -2, for remainders.

The quotients being x, y, and z,

$$V = 3x + 1 = 7y + 4 = 9z - 2$$

$$3x + 1 = 7y + 4, \text{ and hence } x = 2y + 1 + \frac{y}{3};$$

then if $\frac{y}{3} = r$, $y = 3r$.

Hence $7y + 4 = 9z - 2$ becomes $21r + 4 = 9z - 2$,

or $$9z - 21r = 6, \text{ or } 3z - 7r = 2,\ z = 2r + \frac{r+2}{3};$$

and if $\frac{r+2}{3} = s$, $r = 3s - 2$, and $z = 7s - 4$,

and hence $$V = 9z - 2 = 63s - 38,$$

and when $s = 1$, $V = 25 =$ the required number.

658. The number found by the process described in Articles (655) and (656) fulfils the required conditions. For if there be four conditions to be fulfilled, then the least value assigned to p'' that will make V in [7] positive, will evidently afford the least value of V, and this value will afford such values of p' and x''' as will fulfil the equation [6], and consequently the equivalent equation $d''x'' + r'' = d'''x''' + r'''$; and since V in [7], or $d'''x''' + r'''$, is rendered positive by the assigned value of p'', therefore $d''x'' + r''$ is also positive; that is, the value obtained for p' renders V in [5], or $a''d''p' + c''$, positive.

It is similarly proved that the obtained value of p' affords such values to p and x'' as fulfil the equation [4], and also render positive V in [3].

Proceeding in this manner, it appears, whatever are the number of conditions, that the value of V thus obtained is the least possible value that can fulfil the conditions of the question.

EXERCISES.

1. Find the least positive integer, which, being divided by 3, 5, and 6, leaves the remainders 1, 3, and 4, ... = 28.

2. What number, when divided by 5, 7, and 9, leaves the remainders 1, 1, and 0? . . . . = 36.

3. What is the least integer that, being divided by 6, 8, and 9, leaves the remainders 3, − 3, and 0? . . . . = 45.

4. Find the least integer that is divisible by the nine digits, ... = 2520.

5. Find the least number which, being divided by 2, 3, 5, 7, and 11, shall leave the remainders 1, 2, 3, 4, and 5, . . = 1523.

6. The number of leaves in a book is known to be between 200 and 300, and in counting them over by 6 and 6, the remainder is 4; by 8 and 8, the remainder is 2; and by 9 and 9, it is found that there is a deficiency of 2 to make up the last 9: what is the number of leaves? . . . . = 250.

EXPONENTIAL EQUATIONS.

659. In an exponential equation containing only one unknown quantity, this quantity is an exponent in one or more of the terms.

660. I. Let the equation be $a^x = b$.

Then $x \,.\, \log. a = \log. b$, and $x = \dfrac{\log. b}{\log. a}$,

and as a and b are known quantities, their logarithms can be found; and hence x can be found.

EXAMPLE.

Find the value of x in the equation $25^x = 34{\cdot}56$.

Here $a = 25$, $b = 34{\cdot}56$; and hence

$$x = \frac{\log. b}{\log. a} = \frac{\log. 34{\cdot}56}{\log. 25} = \frac{1{\cdot}53857}{1{\cdot}39794} = 1{\cdot}10062.$$

EXERCISES.

1. Given $75^x = 48713{\cdot}8$, to find x, ... $x = 2{\cdot}5$.
2. Find x in the equation $100^x = 250$ . . $x = 1{\cdot}19897$.

661. II. Let the equation be $a^{c^x} = d$.

Assume $c^x = z$, then $a^z = d$, and hence find z in this last equation by the preceding case; then z is known in the equation $c^x = z$; and c being also known, x can be found by the first case.

662. III. If the equation were $ab^x = cd^x$,

then $$\log. a + x \,.\, \log. b = \log. c + x \,.\, \log. d,$$

and hence $$x = \frac{\log. c - \log. a}{\log. b - \log. d}.$$

663. IV. Let the equation be $x^x = a$,

$$\text{or } x \,.\, \log. x = \log. a \quad \ldots \quad [1].$$

Find by trial two approximate values of x, and denote them by x' and x''; and when they are substituted for x in the given equation, let the results be a' and a''. Then $x \,.\, \log. x = \log. a$, $x' \,.\, \log. x' = \log. a'$, $x'' \,.\, \log. x'' = \log. a''$; and since $\log. x$, $\log. x'$, and $\log. x''$, are nearly equal, therefore the differences $x - x'$,

$x' - x''$, are nearly proportional to the differences log. a — log. a', log. a' — log. a'', or

$$x - x' : x' - x'' = \log. a - \log. a' : \log. a' - \log. a'', \text{ nearly};$$

hence $$x - x' = \frac{\log. a - \log. a'}{\log. a' - \log. a''}(x' - x'').$$

The quantities in this last expression being all known but x, it can be found; for $x - x'$ is found by the above expression, and adding x' to it, $x - x' + x' = x$.

The value of x thus found is a more approximate value than either x' or x'', and it may now be assumed as a new approximate value to be again denoted by x', and the nearest of the two former approximate values (x', x''), or some assumed one still nearer may be taken and denoted by x''; and the same process being performed in regard to these new values x', x'', a still more approximate value of x may be found.

This last value may again be denoted by x', and another near value by x'', and so on. By thus repeating the same process, more and more approximate values of x can be found.

EXAMPLE.

Given $x^x = 7{\cdot}38699$, to find the value of x.

It is found by trial that the value of x lies between 2 and 3. Hence if $x' = 2$, and $x'' = 3$,

$$x' . \log. x' = 2 . \log. 2 = 2 \times {\cdot}3010300 = {\cdot}6020600 = \log. a'$$

$$x'' . \log. x'' = 3 . \log. 3 = 3 \times {\cdot}4771213 = 1{\cdot}4313639 = \log. a'';$$

also $$\log. 7{\cdot}38699 = {\cdot}8684678 = \log. a;$$

hence $$x - x' = \frac{\log. a - \log. a'}{\log. a' - \log. a''}(x' - x'') = \frac{{\cdot}2664078}{-{\cdot}8293039}$$

$$\times (-1) = 0{\cdot}32,$$

and $$x - x' + x' = x = 0{\cdot}32 + 2 = 2{\cdot}32.$$

This value 2·32 is nearer to x than either 2 or 3; and if it now be denoted by x', and if x'' be assumed $= 2{\cdot}4$, these two new values may be used as the two former; thus,

$$x' . \log. x' = 2{\cdot}32 . \log. 2{\cdot}32 = {\cdot}8479322 = \log. a'$$

$$x'' . \log. x'' = 2{\cdot}4 . \log. 2{\cdot}4 = {\cdot}9125069 = \log. a'',$$

and as before $${\cdot}8684678 = \log. a;$$

hence $$x - x' = \frac{\log. a - \log. a'}{\log. a' - \log. a''}(x' - x'') = \frac{\cdot 0205376}{- \cdot 0645747} \times (-0\cdot 8) = \cdot 0255,$$

and $$x - x' + x' = x = \cdot 0255 + 2\cdot 32 = 2\cdot 3455.$$

This last value may now be assumed $= x'$; and another value nearer than 2·32 or 2·4; as, for instance, 2·346 might be assumed for x'', and the process again repeated, in order to find a still nearer value of x. The preceding value, however, is very nearly correct, the true value carried to four decimal places being 2·3456.

EXERCISES.

1. Given $x^x = 42\cdot 8454$, to find x, . . . $x = 3\cdot 2164$.
2. ... $x^x = 18\cdot 4084$, to find x, . . . $x = 2\cdot 8147$.

COMPOUND INTEREST AND CERTAIN ANNUITIES.

I.—COMPOUND INTEREST.

664. As in compound interest, interest is chargeable not only on the principal, but also on the interest as it falls due, the amount therefore becomes a principal at the beginning of each period of payment.*

Let $p =$ the principal,

$a =$ ... amount,

$t =$ ... time or number of payments,

$r =$ ... amount of £1 for one of the periods of payment,

then is $r =$ £1 + the interest of £1 for one of the periods of payment.

665. The interest of £1 for one period will be the rate per cent. divided by 100. Thus, if the rate per cent. be 5, and the period be 1 year, the interest of £1 for 1 period is $= \frac{5}{100} = \cdot 05$; and therefore $r = 1\cdot 05$. So for a rate of 4 per cent. $r = 1 + \frac{4}{100} = 1 + \cdot 04 = 1\cdot 04$. When the interest is payable half-yearly; that is,

* No definitions are here given of the terms—principal, amount, rate per cent., and present value, as these may be found in the treatise on Arithmetic in Chambers's Educational Course.

when one period is half a year, and the rate is 5 per cent. per annum, or 2·5 for one period, $r = 1 + \frac{2 \cdot 5}{100} = \cdot 1 + \cdot 025 = 1 \cdot 025$. If the rate is 4 per cent., and the period a quarter of a year, $r = 1 + \frac{1}{100} = 1 + \cdot 01 = 1 \cdot 01$; and so on for any other rate or period.

Since r is the amount of £1 for one period, and since any two sums are proportional to their amounts for the same rate and time, if a' be the amount of p pounds for one period, then the following proportion is evidently true—namely,

$$1 : p = r : a';$$

hence $a' = pr$; that is,

666. The amount of any principal for one period is equal to its product by the amount of £1 for the same time.

Therefore $pr.r = pr^2 =$ amount of pr pounds for one period,

or	$pr^2 =$	...	p	...	two periods;		
and similarly	$pr^3 =$	...	p	...	three	...	,
or	$pr^4 =$	...	p	...	four	...	;
and generally	$pr^t =$	...	p	...	t	...	.
Hence	$a = pr^t$	...		...	[1].		

667. Cor. When $p = 1$, $a = r^t$; and hence for $t = 1, 2, 3, 4, \ldots$ a is $= r, r^2, r^3, r^4, \ldots$ that is, r being the amount of £1 for one period; r^2 is its amount for two periods; r^3 for three periods, and so on; and generally r^t is its amount of t periods.

By [1], $$p = \frac{a}{r^t} \quad \ldots \quad [2].$$

Hence, by the principles of logarithms,

By [1], $$La = Lp + tLr \quad \ldots \quad [3].$$

Hence $$Lp = La - tLr \quad \ldots \quad [4].$$

$$Lr = \frac{1}{t}(La - Lp) \quad \ldots \quad [5],$$

$$t = \frac{La - Lp}{Lr} \quad \ldots \quad [6].$$

668. It appears from these formulas, that when any three of the four quantities p, a, r, and t, are given, the fourth can be found.

669. Since the amount of a sum of money p for the time t is $a = pr^t$, therefore p is also the present value of a sum a, which is to be paid at a future time distant from the present by t intervals, and by [2], $p = \frac{a}{r^t}$.

II.—AMOUNT OF CERTAIN ANNUITIES.

670. An *annuity* is a constant sum of money paid at regular intervals; and it is said to be *certain*, when it is independent of accidental circumstances.

671. When the annuity is in *arrears;* that is, when it has remained unpaid beyond the period when it is due.

Let a = the annuity,

m = its amount,

t = the time or the number of payments due,

r = the amount of £1 for one period.

Then since the last annuity is due only at the expiry of the time t, no interest is chargeable upon it, so that its amount is only a; the annuity preceding it has been due for one period, and hence (665) its amount is ar; the annuity preceding this has been due for two periods, hence its amount is ar^2; the amount of the next preceding is evidently ar^3; of the next ar^4; and so on to the first, which has been due for $(t-1)$ periods, and its amount is therefore ar^{t-1}. Hence the amount of all the annuities is

$$= a + ar + ar^2 + ar^3 + \dots ar^{t-1}.$$

But this is just an equirational series; and hence (424) its sum is

$$s = \frac{rz - a}{r - 1} = \frac{r.ar^{t-1} - a}{r-1} = \frac{ar^t - a}{r - 1} = \frac{a(r^t - 1)}{r - 1}.$$

Hence $$m = \frac{a(r^t - 1)}{r - 1} \quad \dots \quad [1];$$

therefore $$a = \frac{m(r-1)}{r^t - 1} \quad \dots \quad [2],$$

and logarithmically,

$$Lm = La + L(r^t - 1) - L(r - 1) \quad \dots \quad [3],$$

$$La = Lm - L(r^t - 1) + L(r - 1) \quad \dots \quad [4],$$

$$t = \{L[a + m(r-1)] - La\} \div Lr \quad \dots \quad [5];$$

hence $$tLr = L\{a + m(r-1)\} - La \quad \dots \quad [6],$$

and $$Lr = \frac{L\{a + m(r-1)\} - La}{t} \quad \dots \quad [7].$$

The derivation of the formulas [3] and [4] is easily effected by taking the logarithm of [1], and is left as an exercise to the student.

672. The *present value* of an annuity is the sum that an annuity

is at present worth, supposing it to begin at the present time, and to continue till some future period.

Let $v =$ the present value of the annuity, then it is evident that, if the annuity is to continue from this date for the time t, v must be such a sum as, if laid out at interest for that time, would just be equal to the amount of the annuity for the time t. But (666) the amount of v for that time is $= vr^t$, and (671) the amount of the annuity for that time is $= \dfrac{a(r^t - 1)}{r - 1}$; and therefore

$$vr^t = \frac{a(r^t - 1)}{r - 1};$$

whence $$v = \frac{a(r^t - 1)}{r^t(r - 1)} \quad \ldots \quad [1],$$

and $$a = \frac{vr^t(r - 1)}{r^t - 1} \quad \ldots \quad [2];$$

and logarithmically,

$$Lv = La + L(r^t - 1) - tLr - L(r - 1) \quad \ldots \quad [3],$$

$$La = Lv - L(r^t - 1) + tLr + L(r - 1) \quad \ldots \quad [4],$$

$$t = \{La - L[a - v(r - 1)]\} \div Lr \quad \ldots \quad [5],$$

$$Lr = \{La - L[a - v(r - 1)]\} \div t \quad \ldots \quad [6].$$

The student can easily derive these logarithmic formulas from [1]. In order to find [5], the value of r^t must first be obtained.

673. An annuity is said to be in *reversion*, when it is to begin at some future time, and to continue for a limited period; it is also called a *deferred* annuity.

The present value of a reversionary annuity is evidently equal to the difference between the present values of two other annuities, both of which begin at the present time, and continue, the one to the beginning, the other to the expiry, of the reversion.

Let $t =$ period of the reversion,

$n =$ ... from the present time to the beginning of the reversion,

then $n + t =$ ... from the present time to the expiry of the reversion.

Also, let v be the present value of the reversion, and v', v'', the present values of two annuities, beginning at the present time, and continuing respectively for the periods n and $n + t$; then (672)

$$v' = \frac{a(r^n - 1)}{r^n(r - 1)}, \text{ and } v'' = \frac{a(r^{n+t} - 1)}{r^{n+t}(r - 1)};$$

hence $$v'' - v' = \frac{a}{r^n(r-1)} \cdot \frac{r^t - 1}{r^t} = \frac{a(r^t - 1)}{r^{n+t}(r-1)},$$

therefore $$v = \frac{a(r^t - 1)}{r^{n+t}(r-1)} \quad \ldots \quad [1],$$

and hence $$a = \frac{vr^{n+t}(r-1)}{r^t - 1} \quad \ldots \quad [2];$$

and logarithmically,

$$Lv = La + L(r^t - 1) - (n+t)Lr - L(r-1) \quad \ldots \quad [3],$$
$$La = Lv - L(r^t - 1) + (n+t)Lr + L(r-1) \quad \ldots \quad [4],$$
$$t = \{La - L[a - r^n(r-1)v]\} \div Lr \quad \ldots \quad [5],$$
$$n = \{La + L(r^t - 1) - Lv - tLr - L(r-1)\} \div Lr \quad \ldots \quad [6].$$

The derivation of these logarithmic formulas from [1] is left as an exercise.

674. An annuity is said to be *perpetual*, or it is called a *perpetuity*, when it is to continue for an indefinitely long period.

In this case the value of an annuity may be found from [1] in (672), by considering t to be infinite; thus—

$$v = \frac{a(r^t - 1)}{r^t(r-1)} = \frac{a}{r-1}\left(1 - \frac{1}{r^t}\right);$$

and when $t = \infty$, r^t is $= \infty$, and $\frac{1}{r^t} = \frac{1}{\infty} = 0$;

hence $$v = \frac{a}{r-1} \quad \ldots \quad \ldots \quad [1],$$

therefore $$a = v(r-1) \quad \ldots \quad \ldots \quad [2],$$

and $$r = \frac{a+v}{v} = 1 + \frac{a}{v} \quad \ldots \quad \ldots \quad [3].$$

The value of v in this case may also be easily obtained, by considering that it must be such a sum as will produce an interest $= a$ during one of the intervals of payment. The interest of £1 for one of these intervals is $= r - 1$; and hence that of v is $= v(r-1)$, and therefore $a = v(r-1)$.

MISCELLANEOUS EXERCISES.

1. A draper bought three pieces of cloth, which together measured 159 yards. The second piece was 15 yards longer than the first, and the third 24 yards longer than the second. What was the length of each piece?

First, 35 yds.; second, 50 yds.; third, 74 yds.

2. It is required to divide the number 99 into five such parts, that the first may exceed the second by 3; be less than the third by 10; greater than the fourth by 9; and less than the fifth by 16, . . The parts are 17, 14, 27, 8, and 33 respectively.

3. What number is that the treble of which, increased by 12, shall as much exceed 54 as that treble is below 144? . = 31.

4. A mercer having cut 19 yards from each of three equal pieces of silk, and 17 from another piece of the same length, found that the remnants taken together were 142 yards. What was the length of each piece? . . . . . . = 54 yds.

5. A courier, who travels 60 miles a day, had been despatched 5 days, when a second was sent to overtake him at the rate of 75 miles a day. In what time will the second accomplish his task? . . . . = 20 days.

6. Two workmen, A and B, were employed together for 50 days, at 5s. per day each. A spent sixpence a day less than B did, and at the end of 50 days he found he had saved twice as much as B, and the expense of two days over. What did each spend per day? . . . A spent 50 pence, and B 56 pence per day.

7. A farmer sold 96 loads of hay to two persons. To the first, one-half, and to the second, one-fourth of what his stack contained. How many loads did the stack contain? . . = 128 loads.

8. A person bought two casks of beer, one of which held exactly three times as much as the other. From each of these he drew four gallons, and then found that there were four times as many gallons remaining in the larger as in the other. How many gallons were there in each at first? . . = 36 and 12 gallons.

9. A fish was caught whose tail weighed 9 lbs., his head weighed as much as his tail and half his body; and his body weighed as much as his head and tail. What did the whole fish weigh? . . . . = 72 lbs.

10. In a mixture of wine and cider, 25 gallons more than the half was wine, and 5 gallons less than one-third of the whole was cider. How many gallons were there of each?

= 85 gallons of wine, and 35 of cider.

11. A and B made a joint-stock of £833, which, after a suc-

cessful speculation produced a clear gain of £153. Of this B had £45 more than A. What did each person contribute to the stock? . . . . . B brought in £539, and A £294.

12. Suppose that for every 10 sheep that a farmer kept he should plough an acre of land, and be allowed one acre of pasture for every 4 sheep. How many sheep may that person keep who farms 700 acres? . . . . = 2000 sheep.

13. What number is that to which if 1, 5, and 13 be severally added, the first sum will be to the second as the second to the third? . . . . = 3.

14. When the price of a bushel of barley wanted but 3d. to be to the price of a bushel of oats as 8 to 5, 9 bushels of oats were received as an equivalent for 4 bushels of barley, and 7s. 6d. in money. What was the price of a bushel of each?

The oats was 2s. 6d. per bushel, and the barley 3s. 9d.

15. A market-woman bought a certain number of eggs at two a penny, and as many at three a penny, and sold them out at the rate of five for twopence; after which she found that instead of making her money again as she had expected, she lost fourpence by them. How many eggs of each sort had she? . = 120 each.

16. A hare, fifty of her leaps before a greyhound, takes four leaps to the greyhound's three; but two of the greyhound's leaps are as much as three of the hare's. How many leaps must the greyhound take to catch the hare? . . . = 300 leaps.

17. A person has two sorts of wine, one worth 20 pence a quart, and the other 12 pence. What proportion must he take of each to mix a quart which will be worth 14 pence?

= $\frac{1}{4}$ of the first, and $\frac{3}{4}$ of the second.

18. A merchant buys a cask of brandy for £48, and sells a quantity exceeding three-fourths of the whole by two gallons at a profit of £25 per cent. He afterwards sells the remainder at such a price as to clear £60 per cent. by the whole transaction; and had he sold the whole quantity at the latter price, he would have gained £175 per cent. Required the number of gallons contained in the cask, ... = 120 gallons.

19. Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6, . = 32 and 24.

20. What two numbers are those whose difference, sum, and product are as the numbers 2, 3, and 5 respectively, = 10 and 2.

21. A farmer with 28 bushels of barley, at 2s. 4d. a bushel, would mix rye at 3s. per bushel, and wheat at 4s. per bushel, so that the whole mixture may consist of 100 bushels, and be worth 3s. 4d. per bushel. How many bushels of rye, and how many of wheat, must be mixed with the barley?

= 20 bushels of rye, and 52 of wheat.

22. A rectangular bowling-green having been measured, it was observed that if it were 5 feet broader, and 4 feet longer, it would contain 116 feet more; but if it were 4 feet broader, and 5 feet longer, it would contain 113 feet more. Required the length and breadth, . . . The length was 12, and the breadth 9 feet.

23. A and B each laid out £300 on the purchase of stock; A into the *three per cents.*, and B into the *four per cents.* The stocks were at such a price that B received £1 interest more than A. When afterwards each of the stocks rose 10 *per cent.*, they sold out, and A received £10 more than B. Required the price at which each of the stocks was purchased,

The 3 *per cents.* at £60, and the 4 *per cents.* at £75.

24. A person had a bag of money, containing moidores and guineas, worth £93; but a servant having robbed him of one-sixth of his moidores, and three-fifths of his guineas, left him only £54, 15s. How many moidores and guineas had he at first? . . . . = 30 moidores, and 50 guineas.

25. A person bought some sheep for £72; and found that, if he had bought 6 more for the same money, he would have paid £1 less for each. How many did he buy, and what was the price of each? . . . . = 18 sheep at £4 each.

26. The plate of a looking-glass is 18 inches by 12, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. Required the width of the frame, = 3 inches.

27. A person bought two pieces of cloth of different sorts, whereof the finer cost 4s. a yard more than the other; for the finer he paid £18, but the coarser, which exceeded the finer in length by 2 yards, cost only £16. How many yards were there in each piece, and what was the price of a yard of each?

= 18 yards at £1 each, and 20 at 16s. each.

28. The joint-stock of two partners, A and B, was £416; A's money was in trade 9 months, and B's 6 months: when they shared stock and gain, A received £228, and B £252. What was each man's stock? . . A's stock was £192, and B's £224.

29. A body of men were formed into a hollow square, three deep, when it was observed that, with the addition of 25 to their number, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. Required the number of men in the hollow square, . . . = [illegible].

30. A vintner sold 7 dozen of sherry and 12 dozen of cl[illegible] for £50; he sold 3 dozen more of sherry for £10 than he did [illegible] for £6. Required the price of each,

A dozen of sherry cost £2, and a dozen of [illegible]

31. The [illegible] makes 6 revolution[illegible]

the hind-wheel in going 120 yards; but if the circumference of each wheel be increased 1 yard, it will make only 4 revolutions more than the hind-wheel in the same space. Required the circumference of each,

The circumference of the less is 4, and of the greater 5 yards.

32. The sum of the squares of the extremes of four numbers in equidifferent progression is 200, and the sum of the squares of the means is 136. What are the numbers? $= \pm 14$, ± 10, ± 6. and ± 2.

33. The sum of the first and second of four numbers in equirational progression is 15, and the sum of the third and fourth is 60. Required the numbers, . . . = 5, 10, 20, and 40.

34. A poulterer bought 15 ducks and 12 turkeys for 5 guineas; he had two ducks more for 18s. than he had of turkeys for 20s. What was the price of each?

The price of a duck was 3s., and of a turkey 5s.

35. There are three numbers in equirational progression, the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers, . . = 3, 6, and 12.

36. There are four numbers in equirational progression, the second of which is less than the fourth by 24, and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers, ... = 1, 3, 9, and 27.

37. There are four numbers in equidifferent progression; the sum of the squares of the first and second is 34, and the sum of the squares of the third and fourth is 130. Required the numbers, ... = 3, 5, 7, and 9.

38. The sum of £700 was divided among four persons, whose shares were in equirational progression; and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares?

= £108, £144, £192, and £256.

39. The sum of four whole numbers in equidifferent progression is 20, and the sum of their reciprocals is $\frac{25}{24}$. Required the numbers, ... = 2, 4, 6, and 8.

40. The three sides of a right-angled triangle are in equidifferent progression, and the difference of the squares of the two sides is 63, and the difference of the squares of the hypotenuse and the least side is 144. Find the three sides of the triangle,

= 9, 12, and 15.

41. The equidifferent mean between two numbers exceeds the equirational by 13, and the equirational exceeds the harmonic by 12. What are the numbers? . . . . = 104, and [illegible]

42. There are four numbers such, that if each be multiplied [illegible] their sum, the products are 252, 504, 396, and 144. Find the [illegible] numbers, ... = 7, 14, 1[illegible]

43. What quantity must be added to each of the terms of the ratio $a:b$, that it may become equal to $c:d$? $= \dfrac{ad-bc}{c-d}$.

44. If the equidifferent mean between a and b be twice as great as the equirational mean; prove that $a:b::2+\sqrt{3}:2-\sqrt{3}$.

45. If the equidifferent mean between a and b be m times the harmonic, then will $\dfrac{a}{b}=\dfrac{\sqrt{m}+\sqrt{(m-1)}}{\sqrt{m}-\sqrt{(m-1)}}$.

46. If the equirational mean between a and b be m times the harmonic, then will $\dfrac{a}{b}=\dfrac{m+\sqrt{(m^2-1)}}{m-\sqrt{(m^2-1)}}$.

47. Prove that in any equirational progression, the sum of the first and last terms is greater than the sum of any other two terms equidistant from the extremes.

48. Prove that in any equirational progression consisting of an even number of terms, the sum of the odd terms is to the sum of the even terms as 1 to the common ratio.

49. The number 2577, expressed in a particular scale of notation, is 40302; find the root of the scale, . . . $= 5$.

50. What is the root of the scale of notation in which a number which is double of 145 will be expressed by the same digits? $= 15$.

51. If n be a whole number, prove that n^3+5n is divisible by 6.

52. Prove that the square of every odd number, diminished by 1, is divisible by 8.

53. If from the cube of any even number there be subtracted 4 times the number itself, the remainder will be divisible by 48. Required a proof.

54. What number, multiplied by 48, will give a product which is a complete fourth power? . . . . . . $= 27$.

55. There are four numbers, the first three of which are in equidifferent, and the last three in harmonic progression; prove that the first has to the second the same ratio which the third has to the fourth.

56. Prove that an equirational mean between two quantities is a mean proportional between an equidifferent and harmonic mean between the same two quantities.

57. Form the equation whose roots are $2+\sqrt{-3}$, $2-\sqrt{-3}$, 1, and -5, ... $= x^4-14x^2+48x-35=0$.

58. Form the equation whose roots are $+\sqrt{-2}$, $-\sqrt{-2}$, 3, and 4, ... $= x^4-7x^3+14x^2-14x+24=0$.

59. Form the equation whose roots are $3+\sqrt{-2}$, $3-\sqrt{-2}$, 2, and 3, . . . $= x^4-11x^3+47x^2-91x+66=0$.

60. Find, by the method of divisors, the roots of the equation, $x^3 - 9x^2 + 22x - 24 = 0$, $= 6$, $\frac{1}{2}(3 + \sqrt{-7})$, and $\frac{1}{2}(3 - \sqrt{-7})$.

61. Find, by the method of divisors, the roots of the equation, $x^4 - 4x^3 - 8x + 32 = 0$, $= 2, 4, -1 + \sqrt{-3}$, and $-1 - \sqrt{-3}$.

62. Find, by the method of divisors, the roots of the equation, $x^4 + x^3 - 29x^2 - 9x + 180 = 0$, $= +3, -3, +4$, and -5.

63. Find, by the method of divisors, the roots of the equation, $x^5 - 17x^4 + 101x^3 - 223x^2 - 18x + 360 = 0$,

$= -1, +3, +4, +5$, and $+6$.

64. Find the least values of x and y that fulfil the conditions of the equation $19x - 117y = 11$, ... $= 56$ and 9.

65. Find all the values of x and y that the following equation admits of in integers, $13x + 14y = 200$, . . $= 10$ and 5.

66. Find two fractions having 5 and 7 for denominators, whose sum is equal to $\frac{26}{35}$, ... $= \frac{3}{5}$ and $\frac{1}{7}$.

67. What number is that which, being divided by 9 and 13, shall leave for remainders 5 and 12? . . . . . $= 77$.

68. A root of the equation, $x^4 + 3x^3 + 2x^2 + 6x - 148 \cdot 6 = 0$, lies between 2 and 3: it is required to find that root,

$= 2 \cdot 7344$ nearly.

69, One root of the equation, $x^5 + 6x^4 - 10x^3 - 112x^2 - 207x - 110 = 0$, lies between 4 and 5. Required that root, $= 4 \cdot 4641016$.

70. One root of the equation, $x^5 + 4x^4 - 2x^3 + 10x^2 - 2x - 962 = 0$, is found to lie between 3 and 4. Required the development of it to seven decimal places, ... $= 3 \cdot 35464$..

THE END.

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